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MA 509 - REAL ANALYSIS - LECTURE 36

Thm. 5.6 Suppose f is differentiable in (a,b).

(i) If $f'(x) \geq 0 \ \forall x \in (a,b)$, then f is monotonically increasing.

(ii) If $f'(x) = 0 \ \forall x \in (a,b)$, then f is constant.

(iii) If $f'(x) \leq 0 \ \forall x \in (a,b)$, then f is monotonically decreasing.

Proof: Let $x_1, x_2 \in (a,b) \ni a < x_1 < x_2 < b$.
Since f is continuous on $[x_1, x_2]$ & differentiable on (x_1, x_2). So $\exists\, x \in (x_1, x_2) \ni$

$$f(x_2) - f(x_1) = (x_2 - x_1) \cdot f'(x). \quad \text{(by MVT)}$$

So for example, if $f'(x) \geq 0$, then $x_2 > x_1$ implies, $f(x_2) \geq f(x_1)$. The other two conclusions follow similarly. ∎

The continuity of derivatives

Consider functions which are differentiable on every point of an interval. Then their derivatives have one important property common with functions continuous on an interval, and that is, intermediate values are assumed.

Thm. 5.7 Suppose f is a real differentiable function on $[a,b]$, and suppose $f'(a) < \lambda < f'(b)$. Then $\exists\, x \in (a,b)$ $\ni f'(x) = \lambda$.

A similar result holds if $f'(a) > f'(b)$.

Proof: Let $g(t) = f(t) - \lambda t$.

Claim: $g(t_1) < g(a)$ for some $t_1 \in (a,b)$ —— (a)

This is because, if $g(t) \geq g(a)$ $\forall$ $t \in (a,b)$, then $\frac{g(t)-g(a)}{t-a} \geq 0$. Since $g'(a)$ exists,

$g'(a+) = g'(a)$. Hence $\lim_{t \to a+} \frac{g(t)-g(a)}{t-a} \geq 0$

$\Rightarrow g'(a) \geq 0$

This contradicts the fact that $g'(a) < 0$, which, in turn, follows from the fact that $f'(a) < \lambda$.

Similarly, $g'(b) > 0$ implies that $\exists$ $t_2 \in (a,b) \ni$ $g(t_2) < g(b)$. —— (b)

Since f is differentiable on $[a,b]$, it is continuous on $[a,b]$. Hence so is g.

Since $[a,b]$ is compact, g attains its minimum on $[a,b]$ at some point $x \in (a,b)$. (from (a) & (b))

Since $g'(x)$ exists on $[a,b]$, it follows from Thm. 5.3 that $g'(x) = 0$.

$\blacksquare$

Cor. 5.8 If f is differentiable on $[a,b]$, then f' cannot have any simple discontinuities on $[a,b]$.

$$L = \lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{1}{3}\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

$$= \frac{1}{3}\lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{6}.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \Rightarrow \frac{x - \sin x}{x^3}$$

Let $x = 3\theta$. Then $L = \lim_{\theta\to 0} \frac{3\theta - \sin(3\theta)}{(3\theta)^3}$

$$= \frac{1}{27}\lim_{\theta\to 0}\frac{3\theta - (3\sin\theta - 4\sin^3\theta)}{\theta^3} = \frac{1}{27}\left(3\lim_{\theta\to 0}\frac{\theta - \sin\theta}{\theta^3} + 4\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)^3\right)$$

$$= \frac{1}{27}(3L + 4) \Rightarrow 27L = 3L + 4$$

$$\Rightarrow 24L = 4 \Rightarrow L = 1/6.$$

L'HOSPITAL'S RULE

Thm. 5.9 Suppose f and g are real and differentiable in (a,b), and $g'(x) \neq 0$ $\forall x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$.

Suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$. —— (1)

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, —— (2)

or if $g(x) \to +\infty$ as $x \to a$, —— (3)

then $\frac{f(x)}{g(x)} \to A$ as $x \to a$. —— (4)

Analogous result holds if $x \to b$, or if $g(x) \to -\infty$ in (4).

Proof: Consider first the case $-\infty \leq A < +\infty$.

Choose $q \in \mathbb{R} \ni A < q$. Then choose $r \in \mathbb{R} \ni A < r < q$.

By (1), $\exists\ c \in (a,b) \ni a < x < c$ implies

(5) $\longrightarrow \dfrac{f'(x)}{g'(x)} < r$. (Why?) For if, $\dfrac{f'(x)}{g'(x)} \geq r$

$\Rightarrow \lim_{x \to a} \dfrac{f'(x)}{g'(x)} \geq r \Rightarrow A \geq r$ $\rightarrow\!\!\!\leftarrow$

If $a < x < y < c$, then by Thm. 5.4, $\exists\ t \in (x,y) \ni$

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r. \quad \text{— (6)}$$

(Why is $g(x) \neq g(y)$ & $g'(t) \neq 0$?)

by Rolle's theorem, for, if $g(x) = g(y)$, $\exists\ t \in (x,y) \ni g'(t) = 0$ $\rightarrow\!\!\!\leftarrow$.

- Suppose (2) holds. Let $x \to a$ in (6), then

$$\frac{f(y)}{g(y)} < r < q \quad (a < y < c) \quad \text{— (7)}$$

Now suppose (3) holds. Fix y in (6) & choose a point $c_1 \in (a,y) \ni g(x) > g(y)$ and $g(x) > 0$ if $(a < x < c_1)$. Multiplying both sides of (6) by $\dfrac{g(x)-g(y)}{g(x)}$, we see that

$$\frac{f(x)-f(y)}{g(x)} < r\left(\frac{g(x)-g(y)}{g(x)}\right)$$

i.e., $$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \quad (a < x < c_1) \quad \text{— (8)}$$

Let $x \to a$ in (8). Since $g(x) \to \infty$ as $x \to a$, $\exists\ c_2 \in (a, c_1) \ni$

$$\frac{f(x)}{g(x)} < r < q \quad (a < x < c_2). \quad \text{— (9)}$$

(7) & (9) imply that for any q, subject only to the condition $A < q$, $\exists\ c_2 \ni \frac{f(x)}{g(x)} < q$ if $a < x < c_2$. — (10)

Similarly, if $-\infty < A \leq +\infty$, and p is chosen so that $p < A$, $\exists\ c_3 \ni p < \frac{f(x)}{g(x)}$ $(a < x < c_3)$ — (11)

Hence (10) & (11) imply, $\lim_{x \to a} \frac{f(x)}{g(x)} = A$.