7 & 9 imply that for any q, subject only to the condition $A < q$, $\exists\ c_2 \ni \frac{f(x)}{g(x)} < q$ if $a < x < c_2$.

$$\text{———} \quad (10)$$

Similarly, if $-\infty < A \leq +\infty$, and p is chosen so that $p < A$, $\exists\ c_3 \ni p < \frac{f(x)}{g(x)}$ $(a < x < c_3)$.

$$\text{———} \quad (11)$$

Hence 10 & 11 imply, $\lim_{x \to a} \frac{f(x)}{g(x)} = A$.

- $\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{1 + \cos x}{1}$ $= \lim_{x \to \infty} 1 + \cos x$ does not exist.

(Not correct)

- $\lim_{x \to \infty} \frac{\tan^{-1} x}{x}$ $\quad f(x) = \tan^{-1} x$

$g(x) = x$

$g(x) \to \infty$ as $x \to \infty$ ← a

$\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ exists because $\lim_{x \to \infty} \frac{\left(\frac{1}{1+x^2}\right)}{1} = 0$.

$\Rightarrow \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = 0$.

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Derivatives of higher order

- In order for $f^{(n)}(x)$ to exist at a point x, $f^{(n-1)}(t)$ must exist in a neighborhood of x (or a one-sided nbhd, if x is an endpoint of the interval on which f is defined), and $f^{(n-1)}$ must be differentiable at x. Since $f^{(n-1)}$ must exist in a nbhd of x, $f^{(n-2)}$ must be differentiable in that nbhd.

Thm. 5.10 (TAYLOR'S THEOREM)

Suppose f is a real function on $[a,b]$, $n \in \mathbb{N}$, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \quad \text{——} \quad (*)$$

(**)

$$\boxed{\text{Then } \exists\, x \in (\alpha, \beta) \ni f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n}$$

Remarks: ① $n=1$ gives mean value theorem.

② The above theorem implies that f can be approximated by a polynomial of degree $n-1$, and that (**) can be used to estimate the error, provided we know bounds of $|f^{(n)}(x)|$.

Proof: ① Define M by
$$f(\beta) = P(\beta) + M(\beta-\alpha)^n \quad \text{—— ①}$$

Let $g(t) = f(t) - P(t) - M(t-\alpha)^n \quad (a \le t \le b)$. —— ②

Claim: $M = \frac{f^{(n)}(x)}{n!}$ for some $x \in (\alpha, \beta)$.

From (*) and ②,

$$g^{(n)}(t) = f^{(n)}(t) - n!\,M \quad (a<t<b).$$

Thus we will be done if we can show that $g^{(n)}(x) = 0$ for some $x \in (\alpha, \beta)$.

Note that for $k = 0, 1, 2, \ldots, n-1$,
$$P^{(k)}(\alpha) = f^{(k)}(\alpha)$$
$$\Rightarrow g(\alpha) = g'(\alpha) = \ldots = g^{(n-1)}(\alpha) = 0.$$

By our choice of M, $g(\beta) = 0$.
Thus by Rolle's thm, $\exists\ x_1 \in (\alpha, \beta) \ni g'(x_1) = 0$.

Again since $g'(\alpha) = 0$, $\exists\ x_2 \in (\alpha, x_1) \ni g''(x_2) = 0$

After n steps, we see that $\exists\ x_n \in (\alpha, x_{n-1}) \ni g^{(n)}(x_n) = 0$.

$\boxtimes$.

Differentiation of Vector-valued functions

- Defn. of derivative same for complex functions f defined on $[a, b]$.
- Same with 'Differentiability $\rightarrow$ Continuity' & rules of differentiation.

If f_1 & f_2 are real and imaginary parts of f, in other words, if

$$f(t) = f_1(t) + i f_2(t), \quad \text{for } a \le t \le b,$$

where $f_1(t)$ & $f_2(t)$ are real, then

$$f'(x) = f_1'(x) + f_2'(x).$$

- f is differentiable iff both f_1 & f_2 are differentiable at x.

* For $\bar{f}: [a,b] \rightarrow \mathbb{R}^k$, $\bar{f}'(x)$ is that point of $\mathbb{R}^k$ for which $\displaystyle\lim_{t \to x} \left| \frac{\bar{f}(t) - \bar{f}(x)}{t - x} - \bar{f}'(x) \right| = 0$.

- The L'Hospital rule as well as the mean value theorem fails to be true for complex-valued functions.

Examples ① Let $x \in \mathbb{R}$. Define $f(x) = e^{ix} = \cos x + i \sin x$

Then $f(2\pi) - f(0) = 1 - 1 = 0$ but $f'(x) = ie^{ix}$

Hence $|f'(x)| = 1 \;\forall\, x \in \mathbb{R}$.

② On $(0,1)$, define $f(x) = x$ & $g(x) = x + x^2 e^{i/x^2}$.

$$\lim_{x\to 0}\left|\frac{f(x)}{g(x)} - 1\right| = \lim_{x\to 0}\left|\frac{x}{x+x^2e^{i/x^2}} - 1\right|$$
$$= \lim_{x\to 0}\frac{\left|-x^2e^{i/x^2}\right|}{\left|x+x^2e^{i/x^2}\right|} = \lim_{x\to 0}\frac{x}{\left|1+xe^{i/x^2}\right|}$$

But by reverse-triangle inequality,

$$\left|1+xe^{i/x^2}\right| \geq \big|1-|x|\big| = |1-x|.$$

$$\Rightarrow \lim_{x\to 0}\left|\frac{f(x)}{g(x)} - 1\right| \leq \lim_{x\to 0}\frac{x}{|1-x|} = 0.$$

$$\Rightarrow \lim_{x\to 0}\left|\frac{f(x)}{g(x)} - 1\right| = 0$$

$$\Rightarrow \lim_{x\to 0}\frac{f(x)}{g(x)} = 1.$$

Next, $g'(x) = 1 + \left(2x - \frac{2i}{x}\right)e^{i/x^2}$ $\quad (0<x<1)$

Thus, $|g'(x)| = \left|2x - \frac{2i}{x}\right| - 1 \geq \frac{2}{x} - 1$

(Hence $g'(x) \neq 0$ for, if $g'(x) = 0$ for some x, we would have $0 \geq \frac{2}{x} - 1 \Rightarrow x \geq 2 \rightarrow \cancel{x}$ since $x \in (0,1)$.)

$$\Rightarrow \left|\frac{f'(x)}{g'(x)}\right| = \frac{1}{|g'(x)|} \leq \frac{1}{\frac{2}{x} - 1} = \frac{x}{2-x}.$$

$$\Rightarrow \lim_{x\to 0}\frac{f'(x)}{g'(x)} = 0.$$