FIELDS

A field is a set F with two operations, addition & multiplication, which satisfy the following "field axioms".

(A) Axioms for addition

(A1) If $x, y \in F$, then $x+y \in F$.

(A2) (Commutativity): $x+y = y+x \ \forall x, y \in F$.

(A3) (Associativity): $(x+y)+z = x+(y+z) \ \forall x, y, z \in F$.

(A4) $\exists$ an element $0 \in F \ni 0+x = x \ \forall x \in F$.

(A5) Given $x \in F$, $\exists$ an element $y \in F \ni x+y=0$.
(This element y is generally denoted by $-x$)

(M) Axioms for multiplication

(M1) If $x, y \in F$, then $xy \in F$

(M2) (Commutativity): $xy = yx \ \forall x, y \in F$

(M3) (Associativity): $(xy)z = x(yz) \ \forall x, y, z \in F$.

(M4) $\exists$ element $1 \in F$, $1 \neq 0 \ni 1x = x \ \forall x \in F$.

(M5) If $x \in F$ and $x \neq 0$, $\exists \ y \in F \ni xy = 1$
(This element y is generally denoted by $1/x$)

(D) Distributivity

$\forall x, y, z \in F$, we have

$$x(y+z) = xy + xz.$$

Thm. 1.2 The axioms for additions imply

(a) If $x+y = x+z$, then $y=z$ (Cancellation Law)

(b) If $x+y = x$, then $y=0$ (Uniqueness of additive identity)

(c) If $x+y=0$, then $y=-x$ (Uniqueness of additive inverse)

(d) $-(-x) = x$

Proof: (a) Note that

since $x+y=x+z$

$y = 0+y = (-x+x)+y = -x+(x+y) = -x+(x+z)$
$= (-x+x)+z = 0+z = z$.

(b) Let $z=0$ in (a).

(c) Let $z=-x$ in (a)

(d) From (A5), we have $-x+x=0$ — (*)
Now replace x by $-x$ in (c). This tells us that
if $-x+y=0$, then $y=-(-x)$.
Along with (*), this gives $-(-x)=x$.

Thm. 1.3 The axioms for multiplication implies
(a) If $x \neq 0$ and $xy = xz$, then $y=z$.
(b) If $x \neq 0$ and $xy = x$, then $y=1$.
(c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.
(d) If $x \neq 0$, then $1/(1/x) = x$.

Proof: Similar to that of Thm. 1.2.

Thm. 1.4 The field axioms imply that for any $x, y, z \in F$,
(a) $0x = 0$
(b) If $x \neq 0$ & $y \neq 0$, then $xy \neq 0$
(c) $(-x)y = -(xy) = x(-y)$.
(d) $(-x)(-y) = xy$

Thm. 1.3 The field axioms imply the following stmts. for any $x, y, z \in F$.

(a) $0x = 0$

(b) If $x \neq 0$ & $y \neq 0$, then $xy = 0$.

(c) $(-x)y = -(xy) = x(-y)$

(d) $(-x)(-y) = xy$

Proof: (a) $0x + 0x = (0+0)x = 0x$. By Thm. 1.2(b), $0x = 0$.

(b) Assume $x \neq 0, y \neq 0$, but $xy = 0$. Then

$$1 = \left(\frac{1}{y}\cdot y\right)\left(\frac{1}{x}\cdot x\right) = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)\cdot(xy) = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)\cdot 0 = 0,$$

a con~~d~~tradiction.

(c)

$$(-x)y + xy = (-x+x)y = 0\cdot y = 0$$

$$\Rightarrow (-x)y = -(xy).$$

Similarly, the second part.

(d) Now by (c) and Thm. 1.2 (d),

$$(-x)(-y) = -(x(-y)) = -(-(xy)) = xy.$$

ORDERED FIELD

An ordered field is a field F which is also an ordered set s.t.

(i) $x + y < x + z$, if $x, y, z \in F$ and $y < z$.

(ii) $xy > 0$ if $x \in F, y \in F$, $x > 0, y > 0$.

e.g. $\mathbb{Q}$ is an ordered field.