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MA 509 - REAL ANALYSIS - LECTURE 40

Ch. 7 - SEQUENCES AND SERIES OF FUNCTIONS

Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions defined on a set E, and suppose that $\{f_n(x)\}_{n=1}^{\infty}$ (the sequence of numbers) converges for every $x \in E$. Then we can define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (x \in E)$$

Then we say $\{f_n\}$ converges on E, and that f is the limit, or the limit function of $\{f_n\}$. We also say "$\{f_n\}$ converges to f pointwise on E".

Similarly, if $\sum_{n=1}^{\infty} f_n(x)$ converges for every $x \in E$, and if we define $f(x) = \sum_{n=1}^{\infty} f_n(x)$ $(x \in E)$, then f is called the sum of series $\sum f_n$.

PROBLEMS WORTH STUDYING

① If the functions f_n are continuous, or differentiable, or integrable, is the same true for $\lim_{n\to\infty} f_n$ (provided, it exists)?

② Relation between f_n' & f' where $f = \lim_{n\to\infty} f_n$.

③ Relation between $\int f_n$ & $\int f$, where $f = \lim_{n\to\infty} f_n$.

Note that saying the limit $f_n \cdot f$ is continuous at x implies $\lim_{t\to x} f(t) = f(x)$.

Hence, to ask whether the limit of a sequence of of continuous functions is continuous is the same as asking whether

$$\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t).$$

Examples

① Let $m, n \in \mathbb{N}$. Let $s_{m,n} = \dfrac{m}{m+n}$.

For every fixed n, $\lim_{m\to\infty} s_{m,n} = \lim_{m\to\infty} \dfrac{1}{1+n/m} = 1$.

$\Rightarrow \lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = \lim_{n\to\infty} 1 = 1$.

But for every fixed m,

$\lim_{n\to\infty} \dfrac{m}{m+n} = 0$ so that $\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = 0$

② Let $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$ $(x \in \mathbb{R}, n \in \mathbb{N}\cup\{0\})$

If $x = 0$, then $f_n(x) = 0$ $\forall n \in \mathbb{N}\cup\{0\}$.

$\Rightarrow \sum_{n=0}^{\infty} f_n(0)$ converges.

For $x \neq 0$, $0 < \dfrac{1}{1+x^2} < 1$, hence $\sum_{n=0}^{\infty} f_n(x)$, being a geometric series, converges to

$$x^2 \left\{ \frac{1}{1 - 1/(1+x^2)} \right\} = 1 + x^2.$$

So define $f(x) = \sum_{n=0}^{\infty} f_n(x) = \begin{cases} 0, & x = 0 \\ 1+x^2, & x \neq 0. \end{cases}$

Note that f is a convergent series of continuous functions.

But f is not continuous at $x=0$ since $\lim_{x \to 0} f(x) = 1+(0)^2 = 1 \neq 0 = f(0)$.

(3) Let $m \in \mathbb{N}$. Set $f_m(x) := \lim_{n\to\infty} \{\cos(m! \pi x)\}^{2n}$

If $m! x \in \mathbb{Z}$, $f_m(x) = \lim_{n \to \infty} \{(-1)^{m! x}\}^{2n} = 1$.

For any other values of x, $-1 \le \cos\theta \le 1$ implies that $f_m(x) = 0$.

We can now define $f(x) = \lim_{m\to\infty} f_m(x)$.

Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $m! x$ is never an integer & hence from above, $f_m(x) = 0$ $\forall m$ so that $f(x) = 0$.

On the other hand, for $x = p/q \in \mathbb{Q}$, $m! x \in \mathbb{Z}$ for $m \geq q$. Hence $f(x) = 1$.

Thus,

$$\lim_{m\to\infty} \lim_{n\to\infty} (\cos(m! \pi x))^{2n} = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

Thus f is an everywhere discontinuous limit function.

Show that f is not Riemann-integrable.

(4) Let $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$ $\quad x \in \mathbb{R}, n \in \mathbb{N}$

Then $f(x) = \lim_{n\to\infty} f_n(x) = 0$ $\quad \left(\because |\sin(nx)| \le 1\right)$

$\Rightarrow f'(x) = 0.$

But $f_n'(x) = \dfrac{n\cos(nx)}{\sqrt{n}} = \sqrt{n}\cos(nx).$

$\Rightarrow f_n' \not\to f'$ since, for example, $f_n'(0) \to \infty$ and $f'(0) = 0.$

(5) Let $f_n(x) = n^2 x(1-x^2)^n$ $\quad (0 \le x \le 1,\ n \in \mathbb{N})$

Let $0 < x \le 1$. For Ch. 3, we know that if $p > 0$ & $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \dfrac{n^\alpha}{(1+p)^n} = 0.$

Let $\alpha = 2$ & $p = \dfrac{x^2}{1-x^2}$. Then

$$\lim_{n\to\infty} \frac{n^2}{\left(1+\frac{x^2}{1-x^2}\right)^n} = \lim_{n\to\infty} n^2(1-x^2)^n = 0$$

Since $\lim_{n\to\infty} x = x$, then $\lim_{n\to\infty} f_n(x) = 0.$

Also, $f_n(0) = 0$, then $\lim_{n\to\infty} f_n(0) = 0 \Rightarrow \lim_{n\to\infty} f_n(x) = 0$ (for $0 \le x \le 1$)

Now $\displaystyle\int_0^1 x(1-x^2)^n\,dx = \frac{1}{2n+2}$

$$\Rightarrow \int_0^1 f_n(x)\,dx = \frac{n^2}{2n+2} \longrightarrow \infty \text{ as } n \to \infty.$$

$$\Rightarrow \infty = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0.$$

Suppose $f_n(x) := nx(1-x^2)^n$.

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2}$$

But $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$.

UNIFORM CONVERGENCE

Defn. A sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly on E to a function f if for every $\varepsilon > 0$ $\exists N \in \mathbb{N} \ni$ $n \geq N$ implies $|f_n(x) - f(x)| \leq \varepsilon$ $\forall x \in E$.

Difference between pointwise and uniform convergence

- If $\{f_n\}$ converges pointwise on E, $\exists$ function f s.t. for every $\varepsilon > 0$ and every $x \in E$, $\exists N \in \mathbb{N}$ depending on ε $\boxed{\text{and on } x}$, such that
$$|f_n(x) - f(x)| \leq \varepsilon \text{ for } n \geq N.$$

If $\{f_n\}$ converges uniformly on E, if given $\varepsilon > 0$, we can find one integer N which will do the above $\forall x \in E$.

- $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of its partial sums, i.e., $\left\{\sum_{i=1}^{n} f_i(x)\right\}$ converges uniformly on E.

Thm. 7.2 Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ $(x \in E)$.

Set $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. Then $f_n \to f$ unif. on E iff $M_n \to 0$ as $n \to \infty$.

Proof: "$\Rightarrow$" given $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni n \geq N, x \in E$, $|f_n(x) - f(x)| \leq \varepsilon$

$\Rightarrow M_n = \sup_{x\in E} |f_n(x) - f(x)| \leq \varepsilon$

Thm. 7.3 (Weierstrass M-test)

Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose $|f_n(x)| \leq M_n$ $(x \in E, n \in \mathbb{N})$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof: Since $\sum M_n < \infty$, given $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni$ $\forall n, m \geq N$, $\sum_{i=n}^{m} M_i \leq \varepsilon$.

But $\left|\sum_{i=n}^{m} f_i(x)\right| \leq \sum_{i=n}^{m} |f_i(x)| \leq \sum_{i=n}^{m} M_i \leq \varepsilon$.

By Cauchy criterion for uniform convergence, i.e. Thm. 7.1, we see that $\sum f_n$ converges uniformly on E. ▨