THE EXTENDED REAL NUMBER SYSTEM

Defn. Extended real number system

$$= \mathbb{R} \cup \{\pm\infty\}.$$

The original order in $\mathbb{R}$ is preserved, and we define $-\infty < x < +\infty \quad \forall x \in \mathbb{R}$.

Remark: The extended real number system does not form a field. (Why?)

Conventions: (a) If $x \in \mathbb{R}$,

$$x + \infty = +\infty, \quad x - \infty = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(b) If $x > 0$, $\quad x \cdot (+\infty) = +\infty \quad$ and $\quad x \cdot (-\infty) = -\infty$.

(c) If $x < 0$, $\quad x \cdot (+\infty) = -\infty \quad$ and $\quad x(-\infty) = +\infty$.

THE COMPLEX FIELD

Defn. A complex number is an ordered pair (a, b) of real numbers. (Thus $(a, b) \neq (b, a)$ if $a \neq b$.)

- If $x = (a, b)$ & $y = (c, d)$ are 2 complex numbers, then $a = c$ and $b = d$ iff $x = y$.

Def.

$$x + y = (a + c, b + d)$$
$$xy = (ac - bd, ad + bc).$$

Thm. 1.8 The above definitions of addition and multiplication turn the set of all complex numbers into a field, with $(0,0)$ and $(1,0)$ being the additive & multiplicative identities respectively.

Proof: Axioms of addition are clearly satisfied. ((A1) – (A5))

Axioms for multiplication (M1)–(M3) are easy to verify. Now

(M4) $1 \cdot x = (1,0)(a,b) = ((a)(1)-(b)(0), (1)(b)+(0)(a))$
$= (a,b) = x.$

(M5) If $x \neq 0$, then $(a,b) \neq (0,0) \Rightarrow$ at least one of the real numbers a and b is non-zero. Hence $a^2+b^2 > 0$ so that $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$.

Now $\frac{1}{x} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$, for,

$$x \cdot \frac{1}{x} = (a,b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0) = 1.$$

(D) Distributivity is easy to verify. ⊠

Thm. 1.9, For $a, b \in \mathbb{R}$, we have
$(a,0)+(b,0) = (a+b,0)$ & $(a,0)(b,0) = (ab,0)$.

Remark: Thus $(a,0)$ has same arithmetic properties as a. We therefore identify $(a,0)$ with a.

Defn. $i = (0,1)$.

Thm 1.10 $i^2 = -1$

Proof: $i^2 = (0,1)(0,1) = (-1,0) = -1$.

Thm 1.11 If a and b are real, $(a,b) = a+bi$.

Proof: $a+bi = (a,0)+(b,0)(0,1) = (a,0)+(0,b)$
$= (a,b)$. ⊠

Def: If $a, b \in \mathbb{R}$, and $z = a+bi$, then $\bar{z} = a - bi$ is called the conjugate of z.

$a = Re(z)$, $b = Im(z)$

Thm. 1.12 If $z, w \in \mathbb{C}$, then

(a) $\overline{z+w} = \bar{z} + \bar{w}$. (b) $\overline{zw} = \bar{z}\,\bar{w}$

(c) $z + \bar{z} = 2 Re(z)$, $z - \bar{z} = 2i Im(z)$

(d) $z\bar{z} \in \mathbb{R}^+$ (except when $z = 0$).

Proof: (a), (b), (c) - Exercise

(d): Let $z = a+bi$, then
$$z\bar{z} = (a+bi)(a-bi) = a^2 - b^2 i^2 = a^2 - (-b^2) = a^2 + b^2 > 0$$
except when $z = 0$.

Defn. Let $z \in \mathbb{C}$. Then $|z| :=$ the non-negative sq. root of $z\bar{z}$, that is, $|z| = \sqrt{z\bar{z}}$.
(from Thm. 1.6 & Thm. 1.12 (d)).

- For $x \in \mathbb{R}$, $|x| = \sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

Thm. 1.13 Let $z, w \in \mathbb{C}$. Then

(a) $|z| > 0$ unless $z = 0$, $|0| = 0$.

(b) $|\bar{z}| = |z|$

(c) $|zw| = |z||w|$

(d) $|Re(z)| \leq |z|$

(e) $|z+w| \leq |z| + |w|$

Proof: (a) From Thm. 1.12 (d), $z\bar{z} \in \mathbb{R}^+$ if $z \neq 0$. Hence
$|z| = \sqrt{z\bar{z}} > 0$. If $z = 0$, $|z| = 0$.

(b) $|a - bi| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$.

(c) If $z = a+bi$, $w = c+di$ for $a, b, c, d \in \mathbb{R}$, then
$|zw|^2 = (a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2 =$

(c) $|zw|^2 = |z|^2|w|^2$ is what we have to prove.

$|zw|^2 = |(a+bi)(c+di)|^2 = |(ac-bd)+(ad+bc)|^2$

$= (ac-bd)^2 + (ad+bc)^2 = (a^2+b^2)(c^2+d^2) = |z|^2|w|^2$

$= (|z||w|)^2 \underset{\text{uniqueness}}{\overset{\text{By}}{\Longrightarrow}} |zw| = |z||w|.$

(d) $a^2 \leq a^2+b^2 \quad (\because b^2 \geq 0)$

$\Rightarrow |a| = \sqrt{a^2} \leq \sqrt{a^2+b^2} \Longrightarrow |Re(z)| \leq |z|.$

(e) Note that $\overline{z\overline{w}} = \overline{z}\,\overline{\overline{w}} = \overline{z}w.$

$\Rightarrow z\overline{w} + \overline{z}w = z\overline{w} + \overline{z\overline{w}} = 2Re(z\overline{w}).$

Hence $|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w})$

$= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w}$

$= |z|^2 + 2Re(z\overline{w}) + |w|^2$

$\leq |z|^2 + 2|z\overline{w}| + |w|^2$

$= |z|^2 + 2|z||w| + |w|^2$

$= (|z|+|w|)^2.$

Now take sq. roots $\boxtimes$

CAUCHY-SCHWARZ-BUNIAKOWSKI INEQUALITY

Thm. 1.14 If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j\overline{b_j}\right|^2 \leq \sum_{j=1}^{n}|a_j|^2 \sum_{j=1}^{n}|b_j|^2.$$

Proof: Let $A = \sum |a_j|^2$ (where $\sum = \sum_{j=1}^{n}$)

$B = \sum |b_j|^2$

$C = \sum a_j\overline{b_j}.$

If $B=0$, then $|b_j|^2 \geq 0$ implies $|b_j|^2 = 0 \;\forall\, 1 \leq j \leq n.$

$\Rightarrow b_1 = \cdots = b_n = 0.$

Then we have $0 \leq 0.$