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MA 509 - Real Analysis - Lecture 7

* CAUCHY-SCHWARZ-BUNIAKOWSKI INEQUALITY

Thm. 1.14 If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \bar{b_j}\right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right) \quad (1)$$

Proof: Let $\sum$ denote the sum $\sum_{j=1}^{n}$.

Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b_j}$.

If $B = 0$ then clearly, $b_1, b_2, \ldots, b_n$ all are zero. Hence (1) clearly holds.

Now let $B > 0$. Then

$$0 \leq \sum |Ba_j - Cb_j|^2$$

$$= \sum (Ba_j - Cb_j)(B\bar{a_j} - \overline{Cb_j})$$

$$= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b_j} - BC \sum \bar{a_j} b_j + |C|^2 \sum |b_j|^2$$

$$= B^2 A - B|C|^2 - B|C|^2 + B|C|^2$$

$$= B(AB - |C|^2)$$

Dividing both sides by B, we are led to

$$AB - |C|^2 \geq 0. \quad \blacksquare$$

EUCLIDEAN SPACES

Let $k \in \mathbb{N}$. Let $\bar{x} = (x_1, x_2, \ldots, x_k)$, where $\underbrace{x_1, x_2, \ldots, x_k}_{\text{coordinates of vector } \bar{x}} \in \mathbb{R}$.

(for $k > 1$)

If $\alpha \in \mathbb{R}$, put

$$\bar{x} + \bar{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$$

and $\alpha \bar{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_k)$.

Now if $R^k = \{ \bar{x} : \bar{x} = (x_1, x_2, \ldots, x_k), k \in \mathbb{N}, x_i \in \mathbb{R}, 1 \leq i \leq n \}$,
then $\bar{x} + \bar{y} \in \mathbb{R}^k$, $\alpha \bar{x} \in \mathbb{R}^k$.

$\Rightarrow$ <u>$\mathbb{R}^k$ is a vector space over $\mathbb{R}$</u> (exercise)

where $\bar{0} = \underbrace{(0, 0, 0, \ldots, 0)}_{k\text{-tuple}}$.

<u>Inner product:</u> $\bar{x} \cdot \bar{y} := \sum_{i=1}^{k} x_i y_i$

<u>Norm:</u> $|\bar{x}| = \sqrt{\bar{x} \cdot \bar{x}} = \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}$.

$\mathbb{R}^k$ with the above inner product and norm is called <u>euclidean k-space</u>.

Thm. 1.15 Suppose $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$. Then:

(a) $|\bar{x}| \geq 0$ $\quad |\bar{x}| = \sqrt{\sum_{i=1}^{k} x_i^2} \geq 0$ (since $x_i^2 \geq 0$, $1 \leq i \leq k$)

(b) $|\bar{x}| = 0$ iff $\bar{x} = 0$

(c) $|\alpha \bar{x}| = |\alpha| |\bar{x}|$ $\quad |\alpha \bar{x}| = |(\alpha x_1, \alpha x_2, \ldots, \alpha x_k)| = \sqrt{\sum_{i=1}^{k} (\alpha x_i)^2}$

(d) $|\bar{x} \cdot \bar{y}| \leq |\bar{x}| |\bar{y}|$

(e) $|\bar{x} + \bar{y}| \leq |\bar{x}| + |\bar{y}|$

(f) $|\bar{x} - \bar{z}| \leq |\bar{x} - \bar{y}| + |\bar{y} - \bar{z}|$

Proof: (a), (b), (c) trivial

(d)

(Cauchy-Schwarz-Buniakowski ineq.)

$$|\bar{x} \cdot \bar{y}| = \left| \sum_{i=1}^{k} x_i y_i \right| \leq \sqrt{\left(\sum_{i=1}^{k} x_i^2\right)\left(\sum_{i=1}^{k} y_i^2\right)}$$

$$= |\bar{x}| |\bar{y}|.$$

(e)

$$\begin{aligned} |\bar{x}+\bar{y}|^2 &= (\bar{x}+\bar{y}) \cdot (\bar{x}+\bar{y}) \\ &= \bar{x} \cdot \bar{x} + 2\bar{x} \cdot \bar{y} + \bar{y} \cdot \bar{y} \\ &\leq |\bar{x}|^2 + 2|\bar{x}| \cdot |\bar{y}| + |\bar{y}|^2 \\ &= (|\bar{x}| + |\bar{y}|)^2. \end{aligned}$$

$\bar{x} \cdot \bar{y} \leq |\bar{x}| |\bar{y}|$ ($\because$ $|a| \leq b \Leftrightarrow -b \leq a \leq b$)

Now take sq. roots on both sides.

(f) In (e), replace $\bar{x}$ by $\bar{x} - \bar{y}$ and $\bar{y}$ by $\bar{y} - \bar{z}$ so that

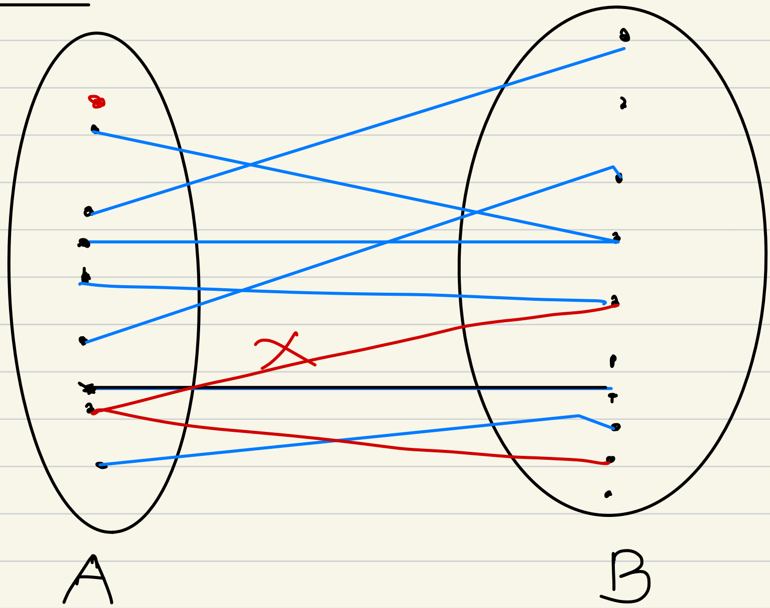
$$|\bar{x} - \bar{z}| = |(\bar{x} - \bar{y}) + (\bar{y} - \bar{z})| \leq |\bar{x} - \bar{y}| + |\bar{y} - \bar{z}|.$$

$\blacksquare$

Remark: (a), (b) and (f) turn $\mathbb{R}^k$ into a metric space.

Chapter 2 — BASIC TOPOLOGY

Function :



Defn: Let A and B be two sets. Suppose that we associate to each element of A, one and only one element of B. Then f is said to be a function from A to B. (denoted by $f: A \to B$).

- A : Domain of f
- Set of all values of f, that is, $\{f(x) : x \in A\}$ is called the range or co-domain of f.

$E \subseteq A$

- $f(E) = \{f(x) : x \in E\}$.

  $f(E)$ is the image of E under f.

$\Rightarrow$ $f(A)$ is the range of f. Note that $f(A) \subseteq B$.

- Surjective function

  f is said to be surjective if $f(A) = B$, that is, to every element $y \in B$, $\exists$ $x \in A \ni f(x) = y$.

In this case, we say f maps A onto B.

- Let $E \subseteq B$. Then

  $f^{-1}(E) = \{x \in A : f(x) \in E\}$.

  $f^{-1}(E)$ is called the inverse image of E under f.

- Injective function

f is said to be injective (one-one) if for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A.

In other words, f is injective if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, where $x_1, x_2 \in A$.

- Bijective function (1-1 correspondence)

f is said to be bijective, if it is injective as well as surjective.

- f is also said to be a 1-1 mapping.

- If 2 sets A and B can be put in a 1-1 correspondence, then we say <u>A and B have the same cardinal number</u> (or A and B have same cardinality) and say $A \sim B$ (A equivalent to B).
  - $A \sim A$ (reflexive)
  - $A \sim B \Leftrightarrow B \sim A$ (symmetric)
  - $A \sim B$ & $B \sim C \Rightarrow A \sim C$ (transitive)

Any relation with the above 3 properties is called an <u>equivalence relation</u>.

Defn. Let $n \in \mathbb{N}$ and $J_n := \{1, 2, \ldots, n\}$.
Let $J = \mathbb{N}$ (the set of all natural numbers.)
For any set A, we say

(a) A is <u>finite</u> if $A \sim J_n$ for some n.
(b) A is <u>infinite</u> if A is not finite
(c) A is <u>countable</u> if $A \sim J$.
(d) A is <u>uncountable</u> if A is neither finite nor countable
(e) A is <u>at most countable</u> if A is finite or countable.