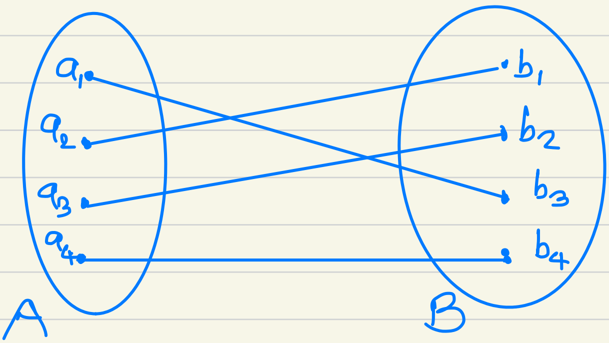
2/9/2020

MA 509 – REAL ANALYSIS – Lecture 8

- If 2 sets A and B can be put in a 1-1 correspondence, then we say A and B have the same cardinal number (or A and B have same cardinality) and say $A \sim B$ (A equivalent to B).

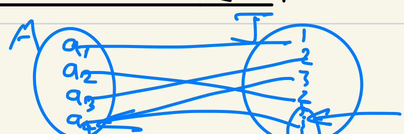
  - $A \sim A$ (reflexive)
  - $A \sim B \Leftrightarrow B \sim A$ (symmetric)
  - $A \sim B$ & $B \sim C \Rightarrow A \sim C$ (transitive)



Any relation with the above 3 properties is called an equivalence relation.

Defn. Let $n \in \mathbb{N}$ and $J_n := \{1, 2, \dots, n\}$.
Let $J = \mathbb{N}$ (the set of all natural numbers)
For any set A, we say

(a) A is finite if $A \sim J_n$ for some n.
(b) A is infinite if A is not finite
(c) A is countable if $A \sim J$.
(d) A is uncountable if A is neither finite nor countable
(e) A is at most countable if A is finite or countable.



Ex. 1 $A = \mathbb{Z}$, $J = \mathbb{N}$.

Then $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even}, \\ -\frac{(n-1)}{2}, & \text{if } n \text{ is odd}, \end{cases}$$

$$\left.\begin{array}{l} f(1) = 0 \\ f(2) = 1 \\ f(3) = -1 \\ f(4) = 2 \\ f(5) = -2 \\ \vdots \end{array}\right.$$

is a bijection, so $A \sim J$, hence countable.

2) Let $f : \mathbb{N} \to 2\mathbb{N}$ defined by
$f(n) = 2n$.
Then check that f is a bijection.

Remark: A finite set cannot be equivalent to one of its proper subsets, i.e; if $E \subsetneq A$, $|A| < \infty$, then $E \nsim A$.

However, this is true for infinite sets as can be seen from the above two examples.

Defn. A sequence is a function defined on the set J of all positive integers.

If $f(n) = x_n$, $n \in J$, then the sequence is denoted by $\{x_n\}$.

- The terms $x_1, x_2, \ldots$ of a sequence need not be distinct.

- Since every countable set is the range of a 1-1 function defined on J, every countable set can be regarded as the range of a sequence of distinct terms.

Thus, elements of any countable set can be "arranged in a sequence".

Thm. 2.1 Every infinite subset of a countable set is countable.

Proof: Suppose $E \subseteq A$ and E is infinite.

Case 1: If $E = A$, there is nothing to prove.

Case 2: $E \subsetneq A$

$\{x_1, x_2, x_3, \ldots\}$

Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. We construct a sequence $\{n_k\}$ as follows:

Let n_1 be the smallest positive integer s.t. $x_{n_1} \in E$. After choosing $n_1, n_2, \ldots, n_{k-1}$ $(k = 2, 3, 4, \ldots)$, let n_k be the smallest integer greater than n_{k-1} s.t. $x_{n_k} \in E$.

$\left\{ E = \underbrace{\{x_{500}, x_{791}, x_{1001}, \ldots\}}_{\text{infinite}} \right\}$

Now let $f(k) = x_{n_k}$, $k \in J$.

This implies that $E \sim J$, whence E is countable.

- No uncountable set can be a subset of a countable set.