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MA 509 - REAL ANALYSIS - LECTURE 9

Defn. Let A & Ω be sets. Suppose that with each element α of A there is associated a subset of Ω, denoted by E_α.

The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$.

Example: $A = \{x : x \in \mathbb{R}, 0 < x \leq 1\} = (0, 1]$.
For every $x \in A$, let $E_x = \{y : y \in \mathbb{R}, 0 < y < x\}$
$= (0, x)$.

Then:

(i) $E_x \subset E_z$ iff $0 < x \leq z \leq 1$.

(ii) $\bigcup_{x \in A} E_x = E_1$.

(iii) $\bigcap_{x \in A} E_x = \phi$.

Proof of (iii): For every $y > 0$, if $x < y$, then $y \notin E_x$.
Hence $y \notin \bigcap_{x \in A} E_x$.

Thm. 2.2 Let $\{E_n\}$, $n \in J$, be a sequence of countable sets. Let $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable.

(In particular, a countable union of countable sets is countable.)

Proof: Let every set E_n be arranged in a seq. $\{x_{nk}\}$, $k = 1, 2, 3, \ldots$. Consider the infinite array

$$\begin{array}{ccccc}
x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\
x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\
x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\
x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\
\cdots & \cdots & \cdots & &
\end{array}$$

For each $n \in J$, the elements of E_n form the n^{th} row. Note that the above array contains all the elements of S.

Arrange the above elements in the sequence

$$x_{11} ; x_{21}, x_{12} ; x_{31}, x_{22}, x_{13} ; x_{41}, x_{32}, x_{23}, x_{14} ; \cdots$$

If any 2 of the sets E_n have elements in common, they will appear more than once in the above sequence.

Hence there is a subset T of the set of natural numbers such that $S \sim T$ so that S is at most countable (by Thm. 2.1).

But $E_1 \subset S$ and E_1 is infinite $\Rightarrow S$ is infinite.

$\Rightarrow S$ is countable.

Remark: A finite union of countable sets is countable.

Cor. 2.3 Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable. Let
$$T = \bigcup_{\alpha \in A} B_\alpha .$$
Then T is at most countable.
In other words, an at most countable union of at most countable sets is at most countable.

Proof: Note that $T \subseteq \bigcup_{n=1}^{\infty} E_n$ from Thm. 2.2. $\blacksquare$

Theorem 2.4 Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \dots, a_n)$, where $a_k \in A$, $1 \le k \le n$, and the elements $a_1, \dots, a_n$ need not be distinct. Then B_n is countable.

Proof: ① B_1 is countable as $B_1 = A$.
② Suppose B_{n-1} is countable, where $n \in J$, $n > 1$.
③ The elements of B_n can be written in the form (b, a), where $b \in B_{n-1}$ and $a \in A$.

Now let b be fixed. Then

$\{(b,a) : b \in B_{n-1} \text{ fixed}, a \in A\} \sim A$, and hence countable.
Thus, B_n is a countable union of countable sets, and hence, by Thm. 2.2, is countable.
$\Rightarrow B_n$ is countable by induction $\forall n \in \mathbb{N}$. $\blacksquare$

Cor. 2.5 The set of all rational numbers is countable.

Proof: Let $n=2$ in Theorem 2.4 and note that (a,b) can be identified with a rational $r = \frac{a}{b}$, $a, b \in \mathbb{Z}$.

$\boxtimes$

Thm. 2.6 Let A be the set of all sequences whose elements are the digits 0 & 1. Then A is uncountable.

Proof: Let $E \subseteq A$ be countable. Let $E = \{s_1, s_2, s_3, \ldots\}$. Note that each s_i is a string consisting of 0's & 1's. We construct a new sequence s as follows:

- If n^{th} digit in s_n is 1, the n^{th} digit of s is 0
- If n^{th} digit in s_n is 0, that in s is 1.

So $s \neq s_n$ $\forall n \in \mathbb{N}$. (as it differs at atleast one place.)

$\Rightarrow s \notin E$ so that $E \subsetneq A$ (i.e., E is a proper subset of A).

Thus, every countable subset of A is a proper subset of A.

$\Rightarrow$ A is uncountable, for if A were countable, the above sentence would imply $A \subsetneq A$, which is absurd.

$\boxtimes$

Remark: This process is called <u>Cantor's diagonalization</u>.

Cor. 2.7 $\mathbb{R}$ is uncountable.

Proof: We assume that every real number has a binary representation (base 2). Then the result follows from the above theorem.

$\boxtimes$

METRIC SPACES

Defn. A metric space X is a set such that with any two elements p, q of X (called points), there is associated a real number $d(p,q)$ (called the distance from p to q) s.t.

(a) $d(p,q) > 0$ if $p \neq q$ & $d(p,p) = 0$.

(b) $d(p,q) = d(q,p)$

(c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

(d is called the <u>distance function</u> or <u>metric</u>.)

Eg: ① $\mathbb{R}$ is a metric space.

② $\mathbb{R}^k$ (Euclidean space) with the metric $d(\bar{x} - \bar{y}) = |\bar{x} - \bar{y}|$, $(x, y \in \mathbb{R}^k)$.

③ $\mathcal{C}(K)$

- Every subset of a metric space is a metric space.