MA 509: Tutorial 10 (2020)

1. [**Continuous Extension Theorem**] Suppose (M_1, ρ_1) is a metric space and A is a dense subset of M_1. If f is a uniformly continuous function from (A, ρ_1) into a complete metric space (M_2, ρ_2), then f can be extended to a uniformly continuous function F from M_1 to M_2.

2. Let f be a real uniformly continuous function on a bounded set E in $\mathbb{R}$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

3. Let f be defined for all real x, and suppose that
$$|f(x) - f(y)| \le (x - y)^2$$
for all real x and y. Prove that f is constant.

4. If
$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$
where $C_0, \cdots, C_n$ are real constants, prove that the equation
$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$
has at least one real root between 0 and 1.

5. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that
$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

6. Suppose g is a real function on $\mathbb{R}$, with bounded derivative (say $|g'| \le M$). Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissible values of ϵ can be determined which depends only on M.)