MA 509: Tutorial 11 (2020)

1. Suppose α increases on $[a,b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.

2. Suppose $f \ge 0$, f is continuous on $[a,b]$, and that $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with the above problem.)

3. Suppose f is a bounded real function on $[a,b]$, and $f^2 \in \mathscr{R}$ on $[a,b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

4. Suppose $f \in \mathscr{R}$ on $[a,b]$ for every $b > a$, where a is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.

Assume that $f(x) \ge 0$ and that f decreases monotonically on $[1,\infty)$. Prove that $\displaystyle\int_1^\infty f(x)\,dx$ converges iff $\displaystyle\sum_{n=1}^\infty f(n)$ converges. (This is the "integral test" for convergence of series.)

5. Let p and q be positive real numbers such that $1/p + 1/q = 1$. Prove the following statements.

(a) [**Young's inequality**] If $u \ge 0$ and $v \ge 0$, then

$$uv \le \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds iff $u^p = v^q$.

(b) If $f \in \mathscr{R}(\alpha), g \in \mathscr{R}(\alpha), f \ge 0, g \ge 0$, and

$$\int_a^b f^p\,d\alpha = 1 = \int_a^b g^q\,d\alpha,$$

then

$$\int_a^b fg\,d\alpha \le 1.$$

(c) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left|\int_a^b fg\,d\alpha\right| \le \left\{\int_a^b |f|^p\,d\alpha\right\}^{1/p} \left\{\int_a^b |g|^p\,d\alpha\right\}^{1/q}.$$

This is *Hölder's inequality*. When $p = q = 2$, it reduces to the Cauchy-Schwarz inequality.

(d) Show that Hölder's inequality is also true for "improper" integrals.

6. Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \le x \le b$. Prove that

$$\int_a^b \alpha(x) g(x)\, dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\, d\alpha.$$

Hint: Take g real, without loss of generality. Given $P = \{x_0, x_1, \cdots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1}) \Delta \alpha_i.$$