14/11/2020

MA 509 – Tutorial 11 solutions

① α increases on $[a,b]$, $a \le x_0 \le b$.
α is continuous at x_0.

$$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{else} \end{cases}$$

To prove: ① $f \in \mathcal{R}(\alpha)$

② $\int f\,d\alpha = 0$.

Proof: ① follows from Thm. 6.7 (of the previous lecture)

② $f \in \mathcal{R}(\alpha) \Rightarrow \int f\,d\alpha$ exists

& $\int f\,d\alpha = \underline{\int} f\,d\alpha = \overline{\int} f\,d\alpha$

Consider a partition P of $[a,b]$.

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i, \text{ where}$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = 0 \quad \text{for } 1 \le i \le n.$$

$$\Rightarrow L(P,f,\alpha) = 0$$

$$\Rightarrow \sup_{\substack{P \text{ is a partition} \\ \text{of } [a,b]}} L(P,f,\alpha) = 0.$$

$$\Rightarrow \underline{\int} f\,d\alpha = 0$$

$$\Rightarrow \int f\,d\alpha = 0.$$

(2) $f \geq 0$, f is continuous on $[a, b]$.

& $\int_a^b f\,dx = 0$.

To prove: $f(x) = 0 \ \forall x \in [a, b]$.

Proof: (by contradiction)

Suppose $\exists x_0 \in [a, b] \ni f(x_0) \neq 0$, so $f(x_0) > 0$.

Since f is continuous on $[a\ b]$,

$\exists \delta > 0 \ni x \in [a, b]$, $|x - x_0| < \delta$ implies $f(x) > \frac{f(x_0)}{2}$, — (*)

(This is because suppose $\forall \delta > 0 \ \exists x \in [a, b] \ni$ $|x - x_0| < \delta$ but $f(x) \leq \frac{f(x_0)}{2}$. Then

$$f(x) - f(x_0) \leq -\frac{f(x_0)}{2} < 0$$

$$\Rightarrow |f(x) - f(x_0)| = -(f(x) - f(x_0)) \geq \frac{f(x_0)}{2}$$

But then this contradicts the fact that f is continuous at x_0.)

Consider the interval $I = [a, b] \cap [x_0 - \delta, x_0 + \delta]$

Then, of course, $I = [c, d]$, where $a \leq c < d \leq b$.

Suppose c, d are elements of a partition P of $[a, b]$, then

$\because f \geq 0$ on $[a, b]$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i \geq \left(\inf_{x \in [c, d]} f(x)\right) \cdot (d - c)$$

$$\geq \frac{f(x_0)}{2}(d - c) > 0$$

from (*)

Thus, $\sup L(P, f) > 0$ —— $(**)$
(P a partition of $[a, b]$)

But $\sup L(P, f) = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$ —— $(***)$

($\because f$ continuous $\Rightarrow$ f is Riemann integrable)

But $\int_a^b f(x)\,dx = 0$ (by hypothesis)

$\Rightarrow$ From $(**)$ & $(***)$, we have

$0 < 0$, a contradiction.

$\Rightarrow f(x) = 0 \quad \forall x \in [a, b]$. ∎

③ f bounded real on $[a, b]$

$f^2 \in \mathcal{R}$ on $[a, b]$

Does $f \in \mathcal{R}$?

Ans. <u>part 1</u> No.

Take $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

$$U(P, f) = \sum_{i=1}^{n} \underline{\underline{M_i}} \Delta x_i$$

$$= \sum_{i=1}^{n} \Delta x_i \quad (\because M_i = 1 \text{ for every } i)$$

$$= x_n - x_0$$

$$= b - a$$

Similarly,

$L(P, f) = a - b$.

$\Rightarrow f \notin \mathcal{R}$.

But $f^2(x) = 1$ on $[a, b]$ & hence $\in \mathcal{R}$.

<u>Part 2</u>

Suppose $f^3 \in \mathcal{R}$.

Take $\varphi(x) = x^{1/3}$ & then use Thm. 6.8 with $h = \varphi \circ f^3$. Then $h = f$.

$\Rightarrow f \in \mathcal{R}$.

* Equality in Young's inequality

Show that the equality in Young's inequality $uv \le \frac{u^p}{p} + \frac{v^q}{q}$ holds iff $u^p = v^q$.

Proof: The equality clearly holds if u or $v=0$. Now assume $u, v \neq 0$.

"$\Leftarrow$" easy. So let $u^p = v^q$.

Note that $\frac{u^p}{p} + \frac{v^q}{q} = v^q\left(\frac{1}{p} + \frac{1}{q}\right) = v^q$

So we are done if we show $u = v^{q-1}$.

Now $u^{1/q} = v^{1/p}$ (from the hypotheses $u^p = v^q$)

$= v^{1-1/q} = v^{\frac{q-1}{q}}$

$\Rightarrow$ Raising both sides to power q, we have

$u = v^{q-1}$

$\boxtimes$

"$\Rightarrow$" To prove this direction, we first prove the following lemma:

Lemma: Let $f: (0, \infty) \to \mathbb{R}$ be defined by
$f(x) = (1 - x^{1-p}) + (p-1)(1-x)$, where $p > 1$.
Then $f(x) = 0$ iff $x = 1$.

Proof:

$$f'(x) = -(1-p)x^{-p} - (p-1)$$
$$= (p-1)(x^{-p} - 1)$$

$f''(x) = -p(p-1)x^{-p-1} < 0$

So f has maximum at that x for which

$f'(x) = 0$. But this happens at $x=1$.
Thus f has a maximum at $x=1$.
But $f(1)=0$.
So that means at any other point x, $f(x)<0$
$\Rightarrow f(x)=0$ iff $x=1$.

Now we prove the other direction.

$$uv = \frac{u^p}{p} + \frac{v^q}{q} = \frac{qu^p + pv^q}{pq}$$

$$= \frac{qu^p + pv^q}{p+q} \qquad \left(\because \frac{1}{p}+\frac{1}{q}=1 \text{ implies } pq = p+q\right)$$

Multiplying both sides by $\frac{(p+q)}{uv}$, we have

$$p+q = q\frac{u^{p-1}}{v} + p\frac{v^{q-1}}{u}$$

$$\Rightarrow q\left(1-\frac{u^{p-1}}{v}\right) = p\left(\frac{v^{q-1}}{u}-1\right)$$

$$\Rightarrow \frac{u^{p-1}}{v} - 1 = \frac{p}{q}\left(1-\frac{v^{q-1}}{u}\right) \quad \text{——} \quad (**)$$

Note that $\frac{u^{p-1}}{v} = \frac{u^{p-1}}{v^{1-p-q+pq}}$ $\quad (\because p+q=pq)$

$$= \frac{u^{p-1}}{v^{(1-p)(1-q)}} = \frac{v^{(1-p)(q-1)}}{u^{1-p}} = \left(\frac{v^{q-1}}{u}\right)^{1-p}.$$

Thus, from $(**)$,

$$\left(\frac{v^{q-1}}{u}\right)^{1-p} - 1 = (p-1)\left(1-\frac{v^{q-1}}{u}\right)$$

$$\left(\because \frac{1}{p}+\frac{1}{q}=1 \Rightarrow \frac{p}{q}=p-1\right)$$

$$\Rightarrow \quad 1-\left(\frac{v^{q-1}}{u}\right)^{1-p} + (p-1)\left(1-\frac{v^{q-1}}{u}\right) = 0$$

Thus $\frac{v^{q-1}}{u}$ is a root of the eq^n. $1-x^{1-p}+(p-1)(1-x)=0$.
Then by the above lemma, $\frac{v^{q-1}}{u}=1$, that is,

$v^{q-1}=u$ so that $uv=v^q$.

But $uv=\frac{u^p}{p}+\frac{v^q}{q}$ implies

$$v^q=\frac{u^p}{p}+\frac{v^q}{q} \Rightarrow v^q\left(1-\frac{1}{q}\right)=\frac{u^p}{p}$$

$$\Rightarrow \frac{v^q}{p}=\frac{u^p}{p}$$

$$\Rightarrow u^p=v^q.$$

$\blacksquare$

(6) Suppose α increases monotonically on $[a,b]$, g is continuous, and $g(x)=G'(x)$ for $a\le x\le b$. Prove that

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b)-G(a)\alpha(a)-\int_a^b G\,d\alpha.$$

Proof:-

W.l.g., let g be real. Let $P=\{x_0,x_1,x_2,\ldots,x_n\}$ be a partition of $[a,b]$, where $x_0=a$ & $x_n=b$.

Since G is differentiable (and hence continuous) on $[a,b]$, by mean value theorem, $\exists\ t_i\in(x_{i-1},x_i) \ni$

$$G'(t_i)=\frac{G(x_i)-G(x_{i-1})}{x_i-x_{i-1}}$$

Since $G'=g$, this gives $g(t_i)\Delta x_i = G(x_i)-G(x_{i-1})$.

$\longleftarrow (*)$

Claim: $\sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1}) \Delta \alpha_i$.

To that end, note that, using (*),

$$\sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i = \sum_{i=1}^{n} \alpha(x_i) \left(G(x_i) - G(x_{i-1}) \right)$$

$$= \sum_{i=1}^{n} \alpha(x_i) G(x_i) - \sum_{i=0}^{n-1} \alpha(x_{i+1}) G(x_i)$$

$$= \alpha(x_n) G(x_n) - \alpha(x_1) G(x_0) + \sum_{i=1}^{n-1} \left(\alpha(x_i) - \alpha(x_{i+1}) \right) G(x_i)$$

$$= G(b)\alpha(b) - G(a)\alpha(x_1) + \sum_{i=0}^{n-1} (\alpha(x_i) - \alpha(x_{i+1})) G(x_i) - (\alpha(x_0) - \alpha(x_1)) G(x_0)$$

$$= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=0}^{n-1} G(x_i) \Delta \alpha_{i+1}$$

$$= G(b) - \alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1}) \Delta \alpha_i .$$

This proves the claim.

Next, note that $\int_a^b G \, d\alpha$ exists since G is continuous on $[a, b]$ (being differentiable on $[a, b]$) and α is monotonic on $[a, b]$.

Thus, given an $\varepsilon > 0$, we can sufficiently refine the partition P to get a partition P^* satisfying

$$U(P^*, G, \alpha) - L(P^*, G, \alpha) < \varepsilon .$$

Thus from Thm. 6.4,

$$\left| \sum_{i=1}^{n} G(x_{i-1}) \Delta \alpha_i - \int_a^b G \, d\alpha \right| < \varepsilon,$$

in particular,

$$\sum_{i=1}^{n} G(x_{i-1}) \Delta \alpha_i > -\varepsilon + \int_a^b G \, d\alpha$$

$$\Rightarrow \sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i$$

$$< G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G \, d\alpha + \varepsilon \quad \text{—— } A$$

Hence,

$$L(P, \alpha g) = \sum_{i=1}^{n} m_i \Delta x_i \quad \Big(\text{where } m_i = \inf_{x \in [x_{i-1}, x_i]} \alpha(x) g(x)\Big)$$

$$\leq \sum_{i=1}^{n} \alpha(t_i) g(t_i) \Delta x_i$$

$$\leq \sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i \quad \left(\begin{array}{l} \because \alpha \text{ increases monotonically on } [a,b], \\ \text{in part. on } [x_{i-1}, x_i], \text{ we have,} \\ \alpha(t_i) \leq \alpha(x_i) \end{array}\right) \quad B$$

From A & B,

$$\Rightarrow L(P, \alpha g) < G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G \, d\alpha + \varepsilon$$

Taking supremum over all partitions P of $[a,b]$, we see that

$$\underline{\int_a^b} \alpha(x) g(x) \, dx \leq G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G \, d\alpha + \varepsilon. \quad \text{—— } 1$$

Also, note that

$$G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G \, d\alpha$$

$$= \sum_{i=1}^{n} \alpha(x_i) g(t_i) \Delta x_i$$

$$= \sum_{i=1}^{n} \alpha(x_i) g(x_i) \Delta x_i - \sum_{i=1}^{n} \alpha(x_i)(g(x_i) - g(t_i)) \Delta x_i$$

$$\leq \sum_{i=1}^{n} M_i^* \Delta x_i + \varepsilon \quad \left(\text{using continuity of } g \text{ on } [a,b]\right)$$

$M_i^* = \sup_{x \in [x_{i-1}, x_i]} \alpha(x) g(x)$

$$= U(P, \alpha g) + \varepsilon$$

Taking infimum on both sides, we see that

$$G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha - \varepsilon \leq \overline{\int_a^b} \alpha(x) g(x)\,dx \quad \text{(2)}$$

Next, we prove the claim.

<u>Claim:</u> $\alpha g \in \mathcal{R}$ on $[a,b]$.

Note that α is monotonically increasing on $[a,b]$. Hence by corollary to Thm. 4.18, α cannot have discontinuities of the second kind.

Hence its only discontinuities are of the first kind, i.e., jumps.

Also, by Thm. 4.19, α being monotonic on $[a,b]$ has at most countably many discontinuities on $[a,b]$, hence αg also has at most countably many discontinuities.

But a function with at most countably many discontinuities is Riemann integrable.

Hence $\underline{\int_a^b} \alpha(x) g(x)\,dx = \int_a^b \alpha(x) g(x)\,dx$

$$= \overline{\int_a^b} \alpha(x) g(x)\,dx \quad \text{(3)}$$

$\Rightarrow$ From ①, ② & ③,

$$-\varepsilon \leq \int_a^b \alpha(x) g(x)\, dx - \left(G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\, d\alpha \right) \leq \varepsilon,$$

that is,

$$\left| \int_a^b \alpha(x) g(x)\, dx - \left(G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\, d\alpha \right) \right| \leq \varepsilon$$

Since ε is arbitrary, we see that

$$\int_a^b \alpha(x) g(x)\, dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\, d\alpha.$$

$\blacksquare$

10] Let p and q be positive real numbers such that $1/p + 1/q = 1$

a) Prove that if $u \geq 0$, $v \geq 0$, then,

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Proof :- Let $f(u) = uv - \frac{u^p}{p}$

Differentiating w.r.t. u,

$$f'(u) = v - \frac{pu^{p-1}}{p}$$

$$\therefore f'(u) = v - u^{p-1}$$

and $f''(u) = -(p-1)u^{(p-2)}$

Now since $p > 0$, $q > 0$, and $\frac{1}{p} + \frac{1}{q} = 1$, we must have $p > 1$, (because for eg. suppose $p = 1/2$, then $\frac{1}{q} = 1 - 2 = -1$; So q becomes negative)

~~Hence we must have~~

This can also be said as follows :-

if $0 < p \leq 1$, then $\frac{1}{p} \geq 1$

$$\therefore \frac{1}{p} + \frac{1}{q} \leq \frac{1}{p}$$

i.e. $\frac{1}{q} \leq 0$

Now $\frac{1}{q} \nless 0$ because q is positive.

Also $\frac{1}{q} \neq 0$ because then $q = \infty$

Hence we cannot have $0 < p \leq 1$

Thus $\underline{p > 1}$ (since p is given to be positive)

$$\therefore f''(u) = -(p-1)u^{p-2} < 0 \quad (\because p > 1)$$

This means that $f(u)$ has maximum at u where $f'(u) = 0$

and $f'(u) = 0 \Rightarrow$

$v - u^{p-1} = 0$

i.e. $u = v^{\frac{1}{p-1}}$

Thus f has maximum at $u = v^{\frac{1}{p-1}}$

$$uv - \frac{u^p}{p} \leq v^{\frac{1}{p-1}} \cdot v - \frac{\left(v^{\frac{1}{p-1}}\right)^p}{p}$$

$$= v^{1+\frac{1}{p-1}} - \frac{v^{\frac{p}{p-1}}}{p}$$

$$= v^{p/p-1} - \frac{v^{p/p-1}}{p}$$

$$= \left(1 - \frac{1}{p}\right) v^{p/p-1}$$

Now $\frac{1}{p} + \frac{1}{q} = 1$

$\therefore \quad \frac{1}{q} = 1 - \frac{1}{p}$

and also $\frac{p}{p-1} = q$.

$\therefore \quad uv - \frac{u^p}{p} \leq \frac{1}{q} v^q$

$\therefore \quad \boxed{uv \leq \frac{u^p}{p} + \frac{v^q}{q}}$

Hence proved.

(b) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$ and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha .$$

then $\int_a^b fg \, d\alpha \leq 1$

Proof:- Since $f \geq 0$, $g \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ (where p, q are positive real nos.) from part (a), we have,

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

$$\therefore \int_a^b fg \, d\alpha \leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha$$

(from Thm. 6.12 part(b), If $f_1(x) \leq f_2(x)$) then $\int_a^b f_1(x)\,dx \leq \int_a^b f_2(x)\,dx$)

$$\therefore \int_a^b fg \, d\alpha \leq \frac{1}{p} \int_a^b f^p \, d\alpha + \frac{1}{q} \int_a^b g^q \, d\alpha$$

$$= \frac{1}{p}(1) + \frac{1}{q}(1) \quad \left(\because \int_a^b f^p \, d\alpha = \int_a^b g^q \, d\alpha = 1 \right)$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

$$\therefore \int_a^b fg \, d\alpha \leq 1$$

Hence proved

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16/2/05

Holder's inequality

* If f and g are complex functions

$$\left| \int_a^b fg\,d\alpha \right| \le \left\{ \int_a^b |f|^p\,d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q\,d\alpha \right\}^{1/q}$$

Proof: We know that

$$\left| \int_a^b fg\,d\alpha \right| \le \int_a^b |f||g|\,d\alpha$$

Now we have to prove that

$$\int_a^b |f||g|\,d\alpha \le \left\{ \int_a^b |f|^p\,d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q\,d\alpha \right\}^{1/q}$$

(Then we will be done)

i.e.

$$\frac{\int_a^b |f||g|\,d\alpha}{\left\{ \int_a^b |f|^p\,d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q\,d\alpha \right\}^{1/q}} \le 1$$

i.e.

$$\int_a^b \left\{ \frac{(|f|^p)^{1/p}}{\left(\int_a^b |f|^p\,d\alpha\right)^{1/p}} \right\} \left\{ \frac{(|g|^q)^{1/q}}{\left(\int_a^b |g|^q\,d\alpha\right)^{1/q}} \right\} d\alpha \le 1 \quad \text{— (A)}$$

~~Let~~ Now from part (b) (prob. 10) (Rudin), we have that if $f \geq 0$, $g \geq 0$ and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha, \text{ then,}$$

$$\int_a^b fg \leq 1 \qquad \text{— (I)}$$

Here let $F = \left(\dfrac{|f|^p}{\int_a^b |f|^p \, d\alpha} \right)^{1/p}$ and $G = \left(\dfrac{|g|^q}{\int_a^b |g|^q \, d\alpha} \right)^{1/q}$

Then $\displaystyle\int_a^b F^p \, d\alpha = \int_a^b \left\{ \left(\frac{|f|^p}{\int_a^b |f|^p \, d\alpha} \right)^{1/p} \right\}^p d\alpha$

$$= \int_a^b \frac{|f|^p}{\int_a^b |f|^p \, d\alpha} \, d\alpha$$

$$= \frac{1}{\int_a^b |f|^p \, d\alpha} \cdot \int_a^b |f|^p \, d\alpha$$

$$= 1$$

$$\therefore \int_a^b F^p \, d\alpha = 1$$

Similarly, $\displaystyle\int_a^b G^q \, d\alpha = 1$

$$\therefore \int_a^b F^p \, d\alpha = 1 = \int_a^b G^q \, d\alpha$$

Hence (I) implies $\int_a^b FG\,d\alpha \leq 1$

Hence (A) is proved.

Hence proved.

8] Suppose $f \in \mathcal{R}$ on $[a,b]$ for every $b>a$, where a is fixed. Define $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$ if this limit exists (and is finite). In that case, we say that the integral on the left converges. If it also converges after f has been replaced by $|f|$, it is said to converge absolutely.

Assume $f(x) \geq 0$ and that f decreases monotonically on $[1,\infty)$. Prove that $\int_1^\infty f(x)\,dx$ converges iff $\sum_{n=1}^{\infty} f(n)$ converges.

Proof :-(i) Given :- $\int_1^\infty f(x)\,dx$ converges

To prove :- $\sum_{k=1}^{\infty} f(k)$ converges

Proof :- Since f is non-increasing, whenever

$n \leq x \leq n+1 \qquad (n \in \mathbb{Z})$

we have, $f(n) \geq f(x) \geq f(n+1)$

Now from 6.12 Thm. (b),

$$\int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

$$\therefore f(n)\int_n^{n+1} 1\,dx \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)\int_n^{n+1} dx$$

$$\therefore f(n)\,(1) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)\,(1)$$

$$\therefore f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)$$

Thus for $N \in \mathbb{Z}$, we have,

$$\sum_{n=1}^{N-1} f(n) \geq \sum_{n=1}^{N-1} \int_n^{n+1} f(x)\,dx \geq \sum_{n=1}^{N-1} f(n+1)$$

i.e. $\sum_{n=1}^{N-1} f(n) \geq \int_1^N f(x)\,dx \geq \sum_{k=2}^{N} f(k)$ — (I)

Now if $\int_1^\infty f(x)\,dx$ converges to A, then by (I),

$$\sum_{k=2}^{N} f(k) \leq \int_1^N f(x)\,dx \leq \int_1^\infty f(x)\,dx = A.$$

i.e. $\sum_{k=2}^{N} f(k) \leq A$

which means that partial sums of $\sum_{k=2}^{\infty} f(k)$ ~~are~~ is bounded above.

But $\left\{\sum_{k=2}^{N} f(k)\right\}$ is a monotonically increasing sequence.

But a bounded monotonic sequence converges.

Hence $\sum_{k=2}^{\infty} f(k)$ converges.

Hence $\sum_{k=1}^{\infty} f(k)$ converges too. ($\because f$ is bounded, $f(1)$ is finite (and positive because $f(x) \geq 0$))

Hence proved.

(ii) ~~Given~~ To prove: if $\sum_{n=1}^{\infty} f(n)$ converges, then $\int_1^{\infty} f(x)\,dx$ converges.

Thus by contrapositive, if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} f(n)$ diverges.

Proof :- from (I),

$$\int_1^N f(x)\,dx \le \sum_{n=1}^{N-1} f(n)$$

Letting $N \to \infty$, we have,

$$\lim_{N \to \infty} \int_1^N f(x)\,dx \le \lim_{N \to \infty} \sum_{n=1}^{N-1} f(n)$$

i.e. $\int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} f(n)$

But $\int_1^{\infty} f(x)\,dx$ diverges,

hence $\sum_{n=1}^{\infty} f(n)$ also diverges.

Hence proved