MA 509: Tutorial 2 (2020)

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

2. Fix $b > 1$.
(a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.
(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
(c) If x is real, define

$$B(x) := \{b^t : t \in \mathbb{Q}, t \leq x\}.$$

Prove that when r is rational, we have $b^r = \sup(B(r))$. Hence it makes sense to define $b^x = \sup(B(x))$ for every real x.
(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

3. Suppose $z = a + ib$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.