MA 509 - Tutorial 2 Solutions

① $r \in \mathbb{Q}$, $r \neq 0$, $x \in \overline{\mathbb{Q}}$

<u>Show:</u> $r+x$, rx are irrational

<u>Proof:</u> Suppose $r+x$ and rx are both rational.

Then $r+x = \frac{m}{n}$, $n \neq 0$, $rx = \frac{p}{q}$, $q \neq 0$

and we know that $r = \frac{a}{b}$, $b \neq 0$

Then $r+x-r = x = \frac{m}{n} - \frac{a}{b} = \frac{mb-na}{nb} \in \mathbb{Q}$ ($b \neq 0$)

Similarly, ($\because r \neq 0$) $\frac{rx}{r} = \frac{p/q}{a/b} = \frac{pb}{aq} \in \mathbb{Q}$ ($q \neq 0$)

These give contradictions since $x \in \overline{\mathbb{Q}}$.

$\boxtimes$

② $b > 1$

ⓐ $m, n, p, q \in \mathbb{Z}$, $n > 0$, $q > 0$ &
$r = \frac{m}{n} = \frac{p}{q}$.

<u>To prove:</u> $(b^m)^{1/n} = (b^p)^{1/q}$

$\exists!\ y \in \mathbb{R}^+ \ni y^n = b^m$ — ①

and $z \in \mathbb{R}^+ \ni z^q = b^p$. — ②

From ①, $(y^n)^q = (b^m)^q$

$$\Leftrightarrow y^{nq} = b^{mq}.$$

$$= b^{np} \quad (\because mq = np)$$

$$= (b^p)^n$$

$$= (z^q)^n$$
$$= z^{nq}$$
$\Rightarrow \quad y^{nq} = z^{nq}$

Take $(nq)^{th}$ root on both sides and appeal to the uniqueness.

$\Rightarrow \quad y = z$

$\Rightarrow \quad (b^m)^{1/n} = (b^p)^{1/q}$.

Hence it makes sense to define

$$b^r = (b^m)^{1/n}.$$

$$\left(\text{i.e.; } b^{m/n} = (b^m)^{1/n} \right)$$

(b) Prove: $b^{r+s} = b^r \cdot b^s$ if $r, s \in \mathbb{Q}$.

Proof: Let $r = \frac{m}{n}$, $s = \frac{p}{q}$

$$r + s = \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}$$

$$b^{r+s} = b^{\frac{mq+np}{nq}} = \left(b^{mq+np}\right)^{\frac{1}{nq}}$$

$$= \left(b^{mq} \cdot b^{np}\right)^{\frac{1}{nq}}$$

$$= \left(b^{mq}\right)^{\frac{1}{nq}} \cdot \left(b^{np}\right)^{\frac{1}{nq}} \quad \left(\because (ab)^{1/n} = a^{\frac{1}{n}} b^{\frac{1}{n}}\right)$$

$$= b^{\frac{mq}{nq}} \cdot b^{\frac{np}{nq}}$$

$$= b^{m/n} \, b^{p/q}$$

$$= b^r \cdot b^s.$$

$\blacksquare$

(c) $x \in \mathbb{R}$.

$B(x) := \{ b^t : t \in \mathbb{Q},\ t \leq x \}$.

Prove: $b^r = \sup(B(r))$, where $r \in \mathbb{Q}$.

Proof: Given $b > 1$.

So if $t \leq r$, then $b^t \leq b^r$.

$\Rightarrow$ b^r is an upper bound of $B(r)$

If b^{r_1} is $\sup(B(r))$, when $r_1 < r$

Then take $\frac{r_1 + r}{2} \in \mathbb{Q}$ which lies between r_1 & r.

Then $b^{\frac{r_1+r}{2}} \leq b^r$

$\Rightarrow$ $b^{\frac{r_1+r}{2}} \in B(r)$

$\rightarrow\leftarrow$ $b^{\frac{r_1+r}{2}} > b^{r_1} = \sup(B(r))$

$\rightarrow\leftarrow$.

$\Rightarrow$ $b^r = \sup(B(r))$.

Hence it makes sense to define

$$b^x := \sup(B(x)).$$

(Aside: $\sqrt{2}^{\sqrt{2}}$: Paulo Ribenboim)

(d) Prove: $b^{x+y} = b^x b^y \ \forall x, y \in \mathbb{R}$.

(1) Let $z = b^{t+u}$, $t, u \in \mathbb{Q}$, $t \le x, u \le y$

$\Rightarrow b^t \cdot b^u \le \sup(B(x)) \sup(B(y))$

$\Rightarrow b^{t+u} \le \sup(B(x)) \sup(B(y))$

$\Rightarrow \boxed{\sup(B(x+y)) \le \sup(B(x)) \sup(B(y))}$ — (1)

(2) $b^t \cdot b^u = b^{t+u} \le \sup(B(x+y))$

$\Rightarrow b^t \le \dfrac{\sup(B(x+y))}{b^u}$

$\Rightarrow \sup(B(x)) \le \dfrac{\sup(B(x+y))}{b^u}$

$\Rightarrow b^u \le \dfrac{\sup(B(x+y))}{\sup(B(x))}$

$\Rightarrow \sup(B(y)) \le \dfrac{\sup(B(x+y))}{\sup(B(x))}$

$\Rightarrow \boxed{\sup(B(x)) \sup(B(y)) \le \sup(B(x+y))}$ — (2)

From (1) & (2),

$\sup(B(x+y)) = \sup(B(x)) \sup B(y)$

$\Rightarrow b^{x+y} = b^x \cdot b^y \ \forall x, y \in \mathbb{R}$. ∎

③ $a = \left(\frac{|w|+u}{2}\right)^{1/2}$, $b = \left(\frac{|w|-u}{2}\right)^{1/2}$

$z = a+ib$, $w = u+iv$.

Prove $z^2 = w$, if $v \geq 0$ and $(\bar{z})^2 = w$ if $v < 0$.

Proof:

$$\begin{aligned} z^2 &= (a+ib)^2 \\ &= a^2 + 2abi + i^2b^2 \\ &= (a^2 - b^2) + i(2ab). \end{aligned}$$

- $a^2 - b^2 = \left\{\left(\frac{|w|+u}{2}\right)^{1/2}\right\}^2 - \left\{\left(\frac{|w|-u}{2}\right)^{1/2}\right\}^2$

$$\begin{aligned} &= \frac{|w|+u}{2} - \frac{|w|-u}{2} \\ &= u \end{aligned}$$

- $2ab = 2\left(\frac{|w|+u}{2}\right)^{1/2}\left(\frac{|w|-u}{2}\right)^{1/2}$

$$\begin{aligned} &= 2\left(\left(\frac{|w|+u}{2}\right)\cdot\left(\frac{|w|-u}{2}\right)\right)^{1/2} \\ &= 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2} \\ &= (|w|^2 - u^2)^{1/2} \\ &= (u^2 + v^2 - u^2)^{1/2} \\ &= (v^2)^{1/2} = \sqrt{v^2} = |v| \\ &= \begin{cases} v & \text{if } v \geq 0 \\ -v & \text{if } v < 0. \end{cases} \end{aligned}$$

$\Rightarrow \quad z^2 = w \quad$ if $v \geq 0$

If $v < 0$, $\quad (\bar{z})^2 = w$.

When $w = 0$, i.e; $u = v = 0$, then $z = \bar{z} = 0$.
Hence every non-zero complex number has 2 square roots, which are complex conjugates of each other.