MA 509: Tutorial 4 (2020)

1. A complex number z is said to be *algebraic* if there are integers $a_0, a_1, \cdots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdot + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers if countable.

(Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.)$$

2. Let E^o denote the set of all interior points of a set E. [E^o is called the interior of E.]

(a) Prove that E^o is always open.

(b) Prove that E is open if and only if $E^o = E$.

(c) If $G \subset E$ and G is open, prove that $G \subset E^o$.

(d) Prove that the complement of E^o is the closure of the complement of E.

(e) Do E and $\bar{E}$ always have the same interiors?

(f) Do E and E^o always have the same closures?

3. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

4. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E ? Answer the same question for closed sets in $\mathbb{R}^2$.

5. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q) \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed?