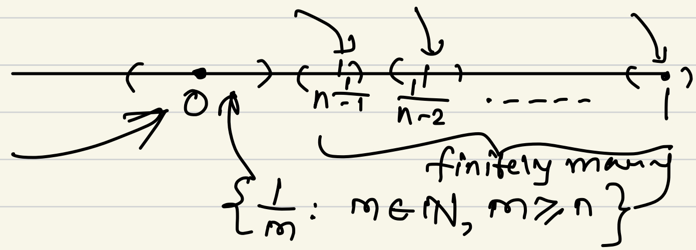
25/9/2020

MA 509 - Tutorial 5 Solutions (2020)

① $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.

Show: K is compact (directly from defn.)

Proof:



② $\{K_\alpha\}$ compact sets with f.i.p. (finite intersection property)

$\Rightarrow \bigcap_\alpha K_\alpha \neq \phi$.

Cor. $\{K_n\}_{n=1}^{\infty}$ compact, $K_n \supset K_{n+1}$ $\forall n \in \mathbb{N}$

$\Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \phi$.

Given the metric space $\mathbb{R}$, give counter-examples to the above result if compact is replaced by closed or bdd.

(1) "Compact" replaced by "bounded":

ⓐ $A_n = [-1/n, 1/n] \setminus \{0\}$ or $(-\frac{1}{n}, \frac{1}{n}) \setminus \{0\}$

$\bigcap_{n=1}^{\infty} A_n = \phi$

ⓑ $A_n = (0, \frac{1}{n})$

$0 < x < 1$

$x > 0$ $\quad nx > 1$

$\therefore x > \frac{1}{n}$ $\quad x \notin (0, \frac{1}{n})$

(c) Let $x \in (0, 1]$. (done previously) from notes

Then for every such x,

$E_x = (0, x)$.

$\bigcap_x^{\infty} E_x = \phi$.

(2) "Compact" replaced by "closed":

$A_n = [n, \infty)$, $n \in \mathbb{N}$

$x > 0$ $\quad$ $n > x$

$x \notin [n, \infty)$

$\bigcap_{n=1}^{\infty} A_n = \phi$.

(3) Consider the metric space $\mathbb{Q}$ with $d(p,q) = |p-q|$, for any $p, q \in \mathbb{R}$.

Let $A_n = \{ p \in \mathbb{Q}^+ : \sqrt{2} - \frac{1}{n} < p < \sqrt{2} + \frac{1}{n} \}$ for each $n \in \mathbb{N}$.

By prob.3 of this tutorial, each A_n is closed in $\mathbb{Q}$.

Also, $A_n \supset A_{n+1}$ $\forall n \in \mathbb{N}$.

Hower $\bigcap_{n=1}^{\infty} A_n = \phi$ since $\sqrt{2} \notin \mathbb{Q}$.

③ $\mathbb{Q}$ metric space with $d(p,q) = |p-q|$.

$E = \{ p \in \mathbb{Q} : 2 < p^2 < 3 \}$.

Show : • E is closed & bounded but not compact.
• Is E open in $\mathbb{Q}$?

Proof: (i) Claim: E is closed, in $\mathbb{Q}$.
We show that E^c is open in $\mathbb{Q}$
Note that
$E^c = E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{ p \in \mathbb{Q} : -\infty < p < -\sqrt{3} \}$$
$$E_2 = \{ p \in \mathbb{Q} : -\sqrt{2} < p < \sqrt{2} \}$$
$$E_3 = \{ p \in \mathbb{Q} : \sqrt{3} < p < \infty \}.$$
(because $\nexists$ rational number whose square is 2 or 3).

Each of E_1, E_2, E_3 can be written in the form $G \cap \mathbb{Q}$, where G is open in X.

for example $E_1 = \underbrace{(-\infty, -\sqrt{3})}_{\text{open interval in } \mathbb{R}} \cap \mathbb{Q}$.

& so on.

$\Rightarrow E^c$ is open in $\mathbb{Q}$ $\Rightarrow E$ is closed in $\mathbb{Q}$.

(ii) Claim: E is bdd in $\mathbb{Q}$, since $-2 < p < 2$.

Claim:

(iii) E is not compact.

1st method:

In any metric space, the following two statements are equivalent:

- E is compact
- Every infinite subset of E has a limit point in E.

Note that if we take S to be a subset of E consisting of rationals tending to $\sqrt{3}$, then the limit point $\sqrt{3} \notin E$

$= \{p \in \mathbb{Q} : 2 < p^2 < 3\}$

$\Rightarrow$ E is not compact.

2nd method: Construct an open cover which doesn't have a finite subcover. for example,

$U_n = \{p : 2 < p^2 < 3 - \frac{1}{n}\}$, $n \in \mathbb{N}$, $n \geq 2$

or $U_n = \{p : 2 + \frac{1}{n} < p^2 < 3\}$, $n \in \mathbb{N}$, $n \geq 2$

Show in each case that U_n is open in $\mathbb{Q}$ for each $n \in \mathbb{N}$, $n \geq 2$. Then show that $\nexists$ finite subcover of $\{U_n\}$.

* Note that E is open in $\mathbb{Q}$ for $E = G \cap \mathbb{Q}$, where

$G := (-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})$ is an open set in $\mathbb{R}$.