**MA 509: Tutorial 6 (2020)**

1. Are closures and interiors of connected sets always connected?
(Hint: Look at subsets of $\mathbb{R}^2$.)

2. (a) If $A$ and $B$ are disjoint closed sets in some metric space $X$, prove that they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix $p \in X, \delta > 0$, define $A$ to be the set of all $q \in X$ for which $d(p, q) < \delta$, define $B$ similarly with $>$ in place of $<$. Prove that $A$ and $B$ are separated.

(d) Prove that every connected metric space with at least two points is uncountable. (Hint: Use (c)).

3. A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.
(Hint: Consider the set of points which have only rational coordinates.)

4. We have proved in class that any non-empty perfect set in $\mathbb{R}^k$ must be uncountable. Imitate its proof (done in class) to obtain the following result:

If $\mathbb{R}^k = \cup_{n=1}^{\infty} F_n$, where each $F_n$ is a closed subset of $\mathbb{R}^k$, then at least one $F_n$ has a non-empty interior.

Equivalent statement: If $G_n$ is a dense open subset of $\mathbb{R}^k$, for $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).

This is a special case of a famous result, namely, the *Baire category theorem*.