23/10/2020

MA 509 - REAL ANALYSIS - TUTORIAL 7

(1) Show that convergence of $\{s_n\}$ implies that of $\{|s_n|\}$. Is the converse true?

$\{s_n\}$ converges, say, to s.

Given $\varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni \forall n \geq N$,

$$|s_n - s| < \varepsilon$$

By reverse-Δ ineq:,

$$\big|\, |s_n| - |s| \,\big| \leq |s_n - s| < \varepsilon$$

$\Rightarrow |s_n| \to |s|$ as $n \to \infty$.

Hence $\{|s_n|\}$ converges.

Converse is NOT true.

Take $\{s_n\} = \{(-1)^n\}$.

$|s_n| = 1$, so $\{|s_n|\}$ converges but $\{s_n\}$ does not.

(2) $$\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n} - n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{(n^2+n) - n^2}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{n}{n\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.$$

(3) $\{s_n\}$ is a real valued seq.
Prove that $\lim_{n\to\infty} s_n = s$ iff

$$\overline{\lim_{n\to\infty}}\, s_n = \underline{\lim_{n\to\infty}}\, s_n = s.$$

Proof: "$\Rightarrow$" $\{s_n\} \to s$ implies any subsequence of $\{s_n\}$ also converges to s.

$\Rightarrow$ the set of subsequential limits, $E, = \{s\}$.
$\sup(E) = s = \inf(E)$.

"$\Leftarrow$" We know that if $x > s$, then $\exists\, N_1 \in \mathbb{N} \ni \forall n \geq N_1$, $s_n < x$

Given $\varepsilon > 0$. Choose the above x to be $s+\varepsilon$ so that $s_n < s+\varepsilon$, $\forall n \geq N_1$.

Similarly, $\exists\, N_2 \in \mathbb{N} \ni \forall n \geq N_2$, $s_n > s-\varepsilon$

Let $N = \max(N_1, N_2)$. Then $\forall n \geq N$,

$$s-\varepsilon < s_n < s+\varepsilon$$
$$\Updownarrow$$
$$|s_n - s| < \varepsilon$$
$$\Rightarrow s_n \to s.$$

$\blacksquare$

(4) Real sequences $\{a_n\}, \{b_n\}$

$$\limsup_{n\to\infty} (a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided RHS is not of the form $\infty - \infty$.

<u>Proof:</u> Assume that $\{a_n\}$ & $\{b_n\}$ are bounded.

Alt. defn. of $\limsup a_n$:

$\{a_n\}$ is bdd. above.

$\left\{ \alpha_k = \sup_{n\geq k} a_n \right\}_{k=1}^{\infty}$ is a decreasing seq.

So it converges to, say, α. Then

$$\alpha = \lim_{k\to\infty} \alpha_k = \inf_{k\geq 1}\left(\sup_{n\geq k} a_n\right).$$

Now $\forall m \geq k$, where k is fixed,

$$\underbrace{a_m + b_m \leq \left(\sup_{n\geq k} a_n\right) + \left(\sup_{n\geq k} b_n\right)}.$$

$$\sup_{m\geq k} (a_m + b_m) \leq \left(\sup_{n\geq k} a_n\right) + \left(\sup_{n\geq k} b_n\right).$$

Now take $k\to\infty$ to get to the reqd. result.

$\boxtimes$

5. $s_1 = 0$, $s_{2m} = \frac{1}{2} s_{2m-1}$, $s_{2m+1} = \frac{1}{2} + s_{2m}$ — (*)

Find: $\limsup_{n\to\infty} s_n$ & $\liminf_{n\to\infty} s_n$.

Sol^n: $0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, \frac{15}{32}, \frac{31}{32}, \ldots$

Claim: $s_{2m} = \frac{1}{2} - \frac{1}{2^m}$

$s_{2m+1} = 1 - \frac{1}{2^m}$

Prove this using (*) & principle of strong math. induction.

This will imply that

$$\limsup_{n\to\infty} s_n = 1 \quad \& \quad \liminf_{n\to\infty} s_n = \frac{1}{2}.$$