MA 509 - Tutorial 8 solutions

5 a $\sum n^3 z^n$

Radius of convergence

$$= \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}, \text{ where } a_n = n^3.$$

$$= \frac{1}{\limsup_{n\to\infty} (n^{1/n})^3} = \frac{1}{1} = 1.$$

b $\sum \frac{2^n z^n}{n!}$

Method 1: Let $b_n = \frac{2^n z^n}{n!}$

Then $\limsup_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \limsup_{n\to\infty} \left| \frac{2^{n+1} z^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n z^n} \right|$

$$= 2|z| \limsup_{n\to\infty} \frac{1}{(n+1)} = 2|z| \cdot 0 = 0 < 1.$$

$\Rightarrow$ by Ratio test, the series converges for all $z \in \mathbb{C}$.

$\Rightarrow$ Radius of convergence $= \infty$.

Method 2: (Root test)

$a_n = \frac{2^n}{n!}$

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left(\frac{2^n}{n!}\right)^{1/n}$$

$$= 2 \limsup_{n\to\infty} \frac{1}{(n!)^{1/n}} .$$

"asymptotic to"

Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

("$f(x) \sim g(x)$" as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$)

$$\Rightarrow \lim_{n\to\infty} \frac{1}{(n!)^{1/n}} = \lim_{n\to\infty} \frac{1}{(2\pi n)^{1/n} \cdot \frac{n}{e}} = 0,$$

$\Rightarrow$ radius of conv. $= \frac{1}{0} = \infty$.

Method 3 : For a seq. of positive numbers $\{a_n\}$, we have

$$0 \leq \limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= 0$$

$$\Rightarrow \limsup_{n\to\infty} |a_n|^{1/n} = 0.$$

(c) $\sum \frac{2^n z^n}{n^2}$ $\quad R = 1/2$

(d) $\sum \frac{n^3 z^n}{3^n}$ $\quad R = 3$

(1) $a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$

Method 1

$\sqrt{n+1} + \sqrt{n} < 2\sqrt{n+1}$

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} > \frac{1}{2\sqrt{n+1}}$$

Since $\sum \frac{1}{\sqrt{n+1}}$ diverges, so does $\sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

Method 2

Consider $\displaystyle\sum_{n=0}^{k} (\sqrt{n+1} - \sqrt{n}) = (\cancel{\sqrt{1}} - \sqrt{0}) + (\cancel{\sqrt{2}} - \cancel{\sqrt{1}}) + (\cancel{\sqrt{3}} - \cancel{\sqrt{2}}) + \cdots + (\cancel{\sqrt{k}} - \cancel{\sqrt{k-1}}) + (\sqrt{k+1} - \cancel{\sqrt{k}})$

$= \sqrt{k+1}$

$$\Rightarrow \lim_{k\to\infty} \sum_{n=0}^{k} (\sqrt{n+1} - \sqrt{n}) = \lim_{k\to\infty} \sqrt{k+1} = \infty.$$

$\Rightarrow \sum a_n$ diverges.

(2) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n} = \dfrac{1}{n(\sqrt{n+1} + \sqrt{n})}$

$\sqrt{n+1} + \sqrt{n} > 2\sqrt{n} \Rightarrow \dfrac{1}{n(\sqrt{n+1}+\sqrt{n})} < \dfrac{1}{2n\sqrt{n}}$

Since $\sum \frac{1}{2n\sqrt{n}}$ converges, by comparison test, so does $\sum a_n$.

$$\frac{\sum a_n + \sum \frac{1}{n^2}}{2} \geq \sqrt{\left(\sum a_n\right)\left(\sum \frac{1}{n^2}\right)}$$

$\geq$

③ $a_n = (n^{1/n} - 1)^n$

Use root test.

Prob.

② Given $a_n \geq 0$

Prove that if $\sum a_n$ converges, so does $\sum \frac{\sqrt{a_n}}{n}$

$\boxed{\text{Method 1}}$ Use A.M. – G.M.

$$\boxed{a = a_n \qquad b = \frac{1}{n^2}}, \text{ where } a_n \geq 0.$$

$$\frac{\sum a_n + \sum \frac{1}{n^2}}{2} \geq \sum \sqrt{\frac{a_n}{n^2}} = \sum \frac{\sqrt{a_n}}{n} \qquad \frac{a+b}{2} \geq \sqrt{ab}$$

$\Rightarrow$ by comparison test, $\sum \frac{\sqrt{a_n}}{n}$ conv.

Method 2: $\sum a_n$ converges $\Rightarrow$ partial

This works only for $a_n \geq 1$ sums of $\sum a_n$ are bounded

But $0 \leq \sqrt{a_n} \leq a_n$ where we take positive sq. root of a_n.

$$\Rightarrow \sum_{n=0}^{k} 0 \leq \sum_{n=0}^{k} \sqrt{a_n} \leq \sum_{n=0}^{k} a_n$$

$\Rightarrow$ partial sums of $\sum \sqrt{a_n}$ are bounded

Also $\left\{\frac{1}{n}\right\}$ is a mon. decr. seq. $\longrightarrow 0$.

$\Rightarrow \sum \frac{\sqrt{a_n}}{n}$ converges.

(1) Investigate the behavior (convergence or divergence) of $\sum a_n$ if $a_n = \frac{1}{1+z^n}$, $z \in \mathbb{C}$.

Ans; <u>Case $|z|>1$:</u> Apply ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1+z^n}{1+z^{n+1}}\right| = \left|\frac{1+\frac{1}{z^n}}{z+\frac{1}{z^n}}\right| \leq \frac{1+\frac{1}{|z|^n}}{\left||z|-\frac{1}{|z|^n}\right|}$$

(by applying triangle inequality for the numerator & the reverse triangle inequality for the denominator.)

$$\Rightarrow \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \leq \limsup_{n\to\infty}\frac{1+1/|z|^n}{\left||z|-\frac{1}{|z|^n}\right|} \quad \text{(Rudin, Thm. 3.19)}$$

$$= \lim_{n\to\infty}\frac{1+1/|z|^n}{\left||z|-\frac{1}{|z|^n}\right|}$$

$$= \frac{1}{|z|} < 1.$$

Thus, by ratio test, $\sum a_n$ converges for $|z|>1$.

<u>Case $|z|\leq 1$:</u> Note that $|1+z^n| \leq 1+|z|^n \leq 2$

$$\Rightarrow |a_n| = \left|\frac{1}{1+z^n}\right| \geq \frac{1}{2}$$

Since $|a_n| \geq 1/2$, $a_n \not\to 0$ as $n\to\infty$.

Hence $\sum a_n$ diverges for $|z|\leq 1$.

5 (p. 79 prob. 8 Rudin)

If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic & bounded, prove that $\sum a_n b_n$ converges.

Proof:- Since $\sum a_n$ converges, its partial sums are bounded. — (1)

Also $\{b_n\}$ is monotonic & bounded, hence converges, say, to b. Without loss of generality, we can assume $\{b_n\}$ to be monotonically decreasing. Note that then $\{b_n - b\}$ is also monotonically decreasing. — (2)

Also if $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} (b_n - b) = 0$ — (3)

From (1), (2) and (3), and Theorem 3.42, $\sum_{n=0}^{\infty} a_n(b_n - b)$ converges.

Since $\sum_{n=0}^{\infty} a_n$ converges, so does $\sum_{n=0}^{\infty} a_n b_n$. ∎

(1) partial sums of $\sum c_n$ form a bdd. seq.

(2) $d_1 \geq d_2 \geq d_3 \geq \text{------}$

(3) $\lim_{n\to\infty} d_n = 0$.

$\Rightarrow \sum c_n d_n$ converges.

Let $d_n = b_n - b$

Since $\{b_n\}$ is monot. decr.

$b_1 - b \geq b_2 - b \geq b_3 - b \geq \dots$

$\Rightarrow d_1 \geq d_2 \geq d_3 \geq \dots$

Use the above thm. with $c_n = a_n$ &

$$d_n = \begin{cases} b_n - b, & \text{if } \{b_n\} \text{ is monot. decr.} \\ b - b_n, & \text{if } \{b_n\} \text{ is monot. incr.} \end{cases}$$

$$\sum a_n(b_n - b) + \sum a_n b = \sum a_n b_n$$

21) Prove that if $\{E_n\}$ is a sequence of closed and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and iff $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.

Proof:- (X,d) is a complete metric space and let $\{E_n\}$ be a sequence of closed sets such that :- i) $E_1 \supset E_2 \supset E_3 \supset \cdots$ and ii) $\lim \operatorname{diam} E_n = 0$

Now for each n, let x_n be an arbitrary point in E_n.

Now since the closed sets form a nested sequence, if $n, m \geq N$, then $x_n, x_m \in E_N$ so that $d(x_n, x_m) \leq \operatorname{diam} E_N$ ($\because \operatorname{diam} E_N = \sup\{d(x_p, x_q) \mid x_p, x_q \in E_N\}$)

Now by hypothesis, N can be chosen sufficiently large so that $\operatorname{diam} E_N < \varepsilon$.

Thus this means that for any $\varepsilon > 0$, there exists an integer N such that for any $n, m \geq N$, $d(x_n, x_m) < \varepsilon$

Thus $\{x_n\}$ is a Cauchy sequence.

Now since X is complete, $\lim x_n$ exists in X, say x_0

$\therefore x_0 = \lim x_n$.

Now from (i), $x_n \in E_N$ for all $n \geq N$.
($\because E_n \in E_N$ for all $n \geq N$)

But E_N is a closed set, so that it contains all its limit points.

$\therefore x_0 \in E_N$. Also since E_{N+1} is closed, $x_0 \in E_{N+1}$ also.
Thus $x_0 \in E_N$ for all N.
But E_N is contained in all E_n for $n \leq N$

This implies, $x_0 \in E_n$ for all $n \in \mathbb{N}$

This implies that
$$x_0 \in \bigcap_{n=1}^{\infty} E_n$$

Let $\bigcap_{n=1}^{\infty} E_n = E$

Now thus E contains at least one point.

If x was another pt. in $E = \bigcap_{n=1}^{\infty} E_n$, then $diam(E) > 0$
But $E_n \supset E \ \forall n \in \mathbb{N} \Rightarrow diam(E_n) \geq diam(E) > 0$
This contradicts the fact that $diam(E_n) \to 0$.

23) Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that sequence $\{d(p_n, q_n)\}$ converges.

Proof :- $\{p_n\}$ is Cauchy sequence in X (metric space)

~~Fix~~

∴ By definition,

for every $\varepsilon > 0$, there exists an integer N, ~~∈ℕ~~ such that $n, m \geq N$, implies,

Fix an $\varepsilon > 0$. Then $\exists$ integer N_1 ~~∈ℕ~~ so that for $n, m \geq N_1$,

$$d(p_n, p_m) < \frac{\varepsilon}{2} \quad \text{— } (1)$$

Also similarly, for this $\varepsilon > 0$, since $\{q_n\}$ is a Cauchy sequence, there exists an integer N_2 ~~∈ℕ~~, Such that

$$d(q_n, q_m) < \frac{\varepsilon}{2} \quad \text{— } (2)$$

Thus for $n, m \geq \max(N_1, N_2) = N$ (say),

$$\left.\begin{aligned} d(p_n, p_m) &< \frac{\varepsilon}{2} \\ \text{and } d(q_n, q_m) &< \frac{\varepsilon}{2}. \end{aligned}\right\} (3)$$

Now by triangle inequality,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_n, q_m)$$

$$\therefore d(p_n - q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_n, q_m)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(from (3))

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i.e. $|d(p_n,q_n) - d(p_m,q_m)| < \varepsilon$

$\forall\ n,m \geq N.$

Thus for every $\varepsilon > 0$, there exist an integer N such that for $n,m \geq N$

$|d(p_n,q_n) - d(p_m,q_m)| < \varepsilon$ ✓

Hence $\{d(p_n,q_n)\}$ is a Cauchy sequence

But it is a Cauchy sequence in $\mathbb{R}$ ✓
$\{\because d(p_n,q_n) \in \mathbb{R}\}$ ✓

✓ And we know that in $\mathbb{R}^k$ (so also in $\mathbb{R}$) every Cauchy sequence converges *since $\mathbb{R}^k$ complete?*

$\therefore$ $\{d(p_n,q_n)\}$ converges. ✓

Hence proved.