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MA 509 – Tutorial 9 solutions

① $f : X \to Y$ is continuous.

To prove: $f(\bar{E}) \subset \overline{f(E)}$.

<u>Proof</u>: Let $y \in f(\bar{E})$. So $\exists\ x \in \bar{E} = E \cup E' \ni$
$y = f(x)$.

<u>Case 1</u>: $x \in E \implies f(x) \in f(E) \subset \overline{f(E)}$.

<u>Case 2</u>: $x \in E'$, that is, x is a limit point of E
So $\exists$ a sequence $\{p_n\}$ in $X \ni p_n \to x$.

But f is continuous on X, in particular, continuous at x.

$\Rightarrow f(p_n) \to f(x) = y$
$\exists$ a sequence in Y, namely, $\{f(p_n)\} \ni$
$f(p_n) \to f(x) = y$.

$\Rightarrow y \in (f(E))' \subset \overline{f(E)}$.

$\Rightarrow f(\bar{E}) \subset \overline{f(E)}$.

- Show that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.

<u>Example</u>:

$f : \underbrace{(0, 1]}_{\text{is a metric space by itself. Call it } E} \to \mathbb{R}$ defined by $f(x) = x$.

$f(\bar{E}) = f((0,1]) = (0,1] \subset [0,1] = \overline{f(E)}$.

(3) f is a cont. real fn. on a metric space X.
$Z(f) = \{p \in X : f(p) = 0\}$
To prove: $Z(f)$ is closed.

<u>Proof:</u> $\{0\}$ is closed.
f is continuous real fn. from X to Y,
$\Rightarrow f^{-1}(\{0\})$ is closed
But $f^{-1}(\{0\}) = Z(f)$.

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<u>2nd method:</u> Let q be a limit point of $Z(f)$.
$\exists \{p_n\}$ in $Z(f) \ni p_n \to q$
$\Rightarrow f(p_n) \to f(q)$, i.e. $\lim_{n\to\infty} f(p_n) = f(q)$
But $\lim_{n\to\infty} f(p_n) = 0 \Rightarrow f(q) = 0 \Rightarrow q \in Z(f)$.

(4) Suppose $f(0) = 0$ or $f(1) = 1$, then we're done.
If not, let $g(x) = f(x) - x$.
Then since f is a function from I to I
and since $f(0) \neq 0$ & $f(1) \neq 1$, we must have
$f(0) > 0$ & $f(1) < 1$, so that
$g(0) > 0$ & $g(1) < 0$
But g is continuous on I. So by intermed-
-iative value theorem, $\exists x \in (0,1) \ni$
$g(x) = 0 \Rightarrow f(x) = x$.

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Take $\varepsilon = 1$. Choose $\delta > 0$. Let $x = x_0 + \min(\delta, 1)$ & $t = \frac{x + x_0}{2}$. Then $|t - x| = \left|\frac{x + x_0}{2} - x\right| = \frac{|x_0 - x|}{2}$

$$= \frac{x - x_0}{2} \leq \frac{\delta}{2} < \delta.$$

But $\left|\frac{1}{t - x_0} - \frac{1}{x - x_0}\right| = \left|\frac{1}{\frac{x + x_0}{2} - x_0} - \frac{1}{x - x_0}\right|$

$$= \left|\frac{1}{x - x_0}\right| = \frac{1}{x - x_0} \geqslant 1 = \varepsilon.$$

Hence $\frac{1}{x - x_0}$ is not uniformly continuous on E.

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4] Let $I = [0,1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Proof :- (by contradiction)

Suppose $f(x) \neq x \quad \forall x \in I$

Therefore construct a function g, such that ~~g(x)x=~~ $g : I \to I$ defined as

$g(x) = f(x) - x \quad \forall x \in I$

Then $g(x) \neq 0 \quad \forall x \in I$. — (1)

Now since f is continuous on I, g ~~x~~ is also continuous (because $\gamma(x) = x$ is a continuous function and so g is the difference of 2 continuous functions f & γ and hence is continuous)

Hence g is continuous on I — (2)

Now $\because g(x) \neq 0$, we must have either $g(x) < 0$ or $g(x) > 0$ i.e. $f(x) < x$ or $f(x) > x$ for any point $x \in I$.

But since f is a mapping from I into I, we cannot have $f(x) < x \quad \forall x \in I$ because then if $x = 0$, then $f(0) < 0$ and we know that f cannot take value less than zero. (because the least value that it may take is 0)

Hence $f(x) < x \quad \forall x \in I$ is not true

So ~~A~~ also $f(x) > x \quad \forall x \in I$ is not true which can be proved ~~easily~~ similarly.

Thus we cannot have $g(x) < 0 \ \forall x \in I$
and also we cannot have $g(x) > 0 \ \forall x \in I$

Hence we must have $g(x) < 0$ for some $x \in I$ and $g(x) > 0$ for remaining values of x in I — (3)

Now from (2), since g is continuous & $I = [0, 1]$ being an interval is connected, $g(I)$ is also connected (since continuous image of connected set is connected)

But then take $p \in I \ni g(p) < 0$ and $q \in I \ni g(q) > 0$.

Now $g(p) < 0 < g(q)$ and $g(x) \neq 0 \ \forall x \in I$ and $g(I)$ is connected.

But this cannot take place because we know that subset E of a real line R is connected if and only if it satisfies the following property:
If $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$.

Hence we must have $0 \in g(I)$
that is there is an $x \in I \ni g(x) = 0$
i.e. $f(x) - x = 0$
i.e. $f(x) = x$.

Hence $f(x) = x$ for at least one $x \in I$
Hence proved.

15] Call a mapping of X into Y open if $f(V)$ is an open set in Y whenever V is an open set in X.

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Proof:- (by contradiction) :-

Suppose f is not monotonic. Then for some $a, b, c \in X \ni a < c < b$ we must have either

(i) $f(a) < f(c)$ and $f(b) < f(c)$ or

(ii) $f(a) > f(c)$ and $f(b) > f(c)$ (because f is neither monotonically increasing nor monotonically decreasing)

If we prove the first case, then the second one can be proved similarly.

(i) $f(a) < f(c)$ and $f(b) < f(c)$ where $a < c < b$

f is defined on R. — (1)

Now consider closed interval $[a, b]$

Since f is continuous and $[a, b]$ is compact (being closed & bounded in R), we have ~~AA~~ that the function f takes its maximum value on some point belonging to $[a, b]$, let that point be x

Thus, $f(y) \leq f(x) \; \forall \; y \in [a, b]$

This x cannot be a because $f(a) < f(c)$

Nor can it be b because $f(b) < f(c)$

$\therefore \; x \in (a, b)$

But since f is open, and (a,b) is an open interval, by defn; $f((a,b))$ is open.

and since ~~f(x)~~ $x \in (a,b)$, $f(x) \in f((a,b))$

Hence since $f(x)$ is an interior point of $f((a,b))$, $\exists\ \delta > 0 \ni$

$f(x) \in (f(x) - \delta, f(x) + \delta) \subset f((a,b))$

from Intermediate Value Th m.

Thus since $f(x) + \frac{\delta}{2} \in f((a,b))$, there is a point $p \in (a,b) \ni f(p) = f(x) + \frac{\delta}{2}$

(by Intermediate value theorem)

But this means that $f(p) > f(x)$

This contradicts the fact that f takes maximum value on x.

Similarly we have contradiction for 2^{nd} case.

Hence our assumption that f is not monotonic is wrong.

Hence f is monotonic.

Hence proved.

3] Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof:- * $Z(f) = \{p : p \in X \text{ and } f(p) = 0\}$ — ①
where f is a continuous real function on a metric space X.

Now we know that finite point set is always compact.
Thus $\{0\}$ is compact.

But $\{0\}$ is a compact subset of metric space ($\mathbb{R}$)
Hence it is closed.

Thus $\{0\}$ is closed in $\mathbb{R}$ and f is a continuous real function on metric space X.

Thus by continuity (corollary of thm. 4.8), $\{f^{-1}(0)\}$ is a closed set in X.

But $\{f^{-1}(0)\}$ is precisely $Z(f)$. (from ①)

Hence $Z(f)$ is closed.

Hence proved.