MA 623: Homework 1 (Due: January 25)

(Note: Justify all the relevant steps.)

1. Prove that if $p \nmid x$, then $p \nmid \binom{p^k x}{p^k}$.

(Note that here x is a positive integer and k is a non-negative integer.)

2. In a letter to Euler in 1742, Goldbach stated that 'Every integer greater than 5 is the sum of three primes'. Euler replied that this was equivalent to 'Every even integer greater than or equal to 4 is the sum of two primes'. Show that these two statements are equivalent.

3. Imitate Euclid's proof to prove that there are infinitely many primes of the form $4n + 3$.

(Hint: Let $p_1, p_2, \ldots, p_k$ be the finite number of such primes (excluding 3). Consider $N = 4p_1 \cdots p_k + 3$. Use the fact that the product of numbers of the form $4k + 1$ is again of the form $4k + 1$ to show that N has a prime divisor of the form $4n + 3$.)

4. Let $F_n = 2^{2^n} + 1$ be the n^{th} Fermat number. Use the identity $a^2 - b^2 = (a - b)(a + b)$ to show that

$$F_n - 2 = F_0 F_1 F_2 \cdots F_{n-1}.$$

Conclude that $(F_n, F_m) = 1$ for all $n \neq m$.