MA 623: Tutorial 2

1. Let $k \in \mathbb{N}$ be fixed. "Evaluate" the following function, that is, write it in the form of simpler arithmetic function(s):

$$\sum_{\substack{d|n \\ k|d}} \mu(d).$$

2. (a) Suppose f is multiplicative. Prove that

$$\sum_{d|n} f(d) = \prod_{p^a||n} \left(1 + f(p) + f(p^2) + \cdots + f(p^a)\right).$$

(Note: Here, the product on the right is taken over all primes p that divide n. The additional information furnished from the notation $p^a||n$ is that the p-adic valuation of n is a.)

(b) Suppose that $\sum_{n=1}^{\infty} |f(n)|$ converges. Show that

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(1 + f(p) + f(p^2) + \cdots\right).$$

3. Assume f is multiplicative. Prove that
(a) $f^{-1}(n) = \mu(n)f(n)$ for square-free n.
(b) $f^{-1}(p^2) = f(p)^2 - f(p^2)$ for every prime p.