MA 631: Special Functions (2022) - HW1

1. For $\mathrm{Re}(\log(a)) > 0$ and $\mathrm{Re}(a) > -1$, evaluate
$$\int_0^\infty x^a a^{-x}\, dx.$$

2. Show that if $z = iy$, where y is real, then
$$|\Gamma(z)| = \sqrt{\frac{\pi}{y \sinh(\pi y)}}.$$
(**Hint:** First prove that $\overline{\Gamma(w)} = \Gamma(\overline{w})$.)

3. Show that
$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2(\theta)}} = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}.$$

4. Show that for $\mathrm{Re}(x) > |\mathrm{Re}(y)|$,
$$\int_0^\infty \frac{\cosh(2yt)\, dt}{\cosh^{2x}(t)} = 2^{2x-2} B(x-y, x+y).$$
(**Hint:** First write the integral over $(-\infty, \infty)$ and write $\cosh(2yt)$ in terms of exponentials. Then make the substitution $u = e^t$.)