26/1/22

MA 631 – SPECIAL FUNCTIONS – Lec. 11

Thm. 17

- Riemann's functional equation for $\zeta(s)$: For all $s \in \mathbb{C}$,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Proof: For $Re(s) > 0$,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} t^{\frac{s}{2}-1} dt$$

Let $t = n^2 \pi x$ so that $dt = n^2 \pi dx$

$$\Rightarrow \Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-n^2\pi x} (n^2 \pi x)^{\frac{s}{2}-1} n^2 \pi dx$$

$$= n^s \pi^{\frac{s}{2}} \int_0^\infty e^{-n^2 \pi x} x^{\frac{s}{2}-1} dx$$

$$\Rightarrow \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1} dx$$

Let $Re(s) > 1$. Then

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n=1}^\infty \int_0^\infty e^{-n^2 \pi x} x^{\frac{s}{2}-1} dx$$

$$= \int_0^\infty \left(\sum_{n=1}^\infty e^{-n^2 \pi x} \right) x^{\frac{s}{2}-1} \, dx$$ (interchange of the order of summation & integration justified by absolute & uniform convergence)

$$= \int_0^\infty \omega(x) \, x^{\frac{s}{2}-1} \, dx, \longrightarrow \text{(a)}$$

where $\omega(x) = \sum_{n=1}^\infty e^{-n^2 \pi x}$.

We note the transformation formula for $\theta(x) = \sum_{n=-\infty}^\infty e^{-n^2 \pi x}$ (theta function):

$$\theta(x^{-1}) = \sqrt{x}\, \theta(x) \quad \text{for } x > 0$$

(This relation can be proved using the Poisson summation formula)

$$\Rightarrow \sum_{n=-\infty}^\infty e^{-\frac{n^2 \pi}{x}} = \sqrt{x} \sum_{n=-\infty}^\infty e^{-n^2 \pi x}$$

$$\Rightarrow 1 + 2\sum_{n=1}^\infty e^{-\frac{n^2 \pi}{x}} = \sqrt{x} \left(1 + 2\sum_{n=1}^\infty e^{-n^2 \pi x} \right)$$

$$\Rightarrow 1 + 2\omega\left(\frac{1}{x}\right) = \sqrt{x}\left(1 + 2\omega(x)\right).$$

$$\Rightarrow \omega\left(\frac{1}{x}\right) = \frac{\left(\sqrt{x} + 2\sqrt{x}\,\omega(x) - 1\right)}{2}$$

$$\Rightarrow \omega\left(\frac{1}{x}\right) = -\frac{1}{2} + \frac{1}{2}\sqrt{x} + \sqrt{x}\,\omega(x) \quad \text{— (b)}$$

From (a),

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty \omega(x)\, x^{\frac{s}{2}-1} dx$$

$$= \int_0^1 \omega(x)\, x^{\frac{s}{2}-1} dx + \int_1^\infty \omega(x)\, x^{\frac{s}{2}-1} dx$$

$$(x \to 1/x)$$

$$= \int_\infty^1 \omega\left(\frac{1}{x}\right) x^{-\frac{s}{2}+1} \left(\frac{-1}{x^2}\right) dx + \int_1^\infty \omega(x)\, x^{\frac{s}{2}-1} dx$$

$$= \int_1^\infty \omega\left(\frac{1}{x}\right) x^{-\frac{s}{2}-1} dx + \int_1^\infty \omega(x)\, x^{\frac{s}{2}-1} dx$$

$$= \int_1^\infty \left(-\frac{1}{2} + \frac{1}{2}\sqrt{x} + \sqrt{x}\,\omega(x)\right) x^{-\frac{s}{2}-1} dx \quad \text{(using (b))}$$

$$+ \int_1^\infty \omega(x)\, x^{\frac{s}{2}-1} dx$$

$$= -\frac{1}{2}\int_1^\infty x^{-\frac{s}{2}-1} dx + \frac{1}{2}\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}} dx$$

$$+ \int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right) dx$$

$$\int_1^\infty x^{-s/2-1} dx = \left[\frac{x^{-s/2}}{-s/2}\right]_1^\infty = -\frac{2}{s}(0-1) = 2/s$$

$$\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\,dx = \left[\frac{x^{-\frac{s}{2}+\frac{1}{2}}}{-\frac{s}{2}+\frac{1}{2}}\right]_1^\infty = \frac{0-1}{-\frac{(s-1)}{2}} = \frac{2}{s-1}.$$

$$\Rightarrow \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$= \frac{-1}{2}\left(\frac{2}{s}\right) + \frac{1}{2}\left(\frac{2}{s-1}\right)$$

$$+ \int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)dx$$

$\Rightarrow$ For $\mathrm{Re}(s) > 1$,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{-1}{s} + \frac{1}{s-1} + \int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)dx$$

$$= \frac{1}{s(s-1)} + \int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)dx \quad \longrightarrow (c)$$

The integral

$\int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)dx$ is an entire function of s because

Claim: $\omega(x) = O(e^{-\pi x})$ as $x \to \infty$.

$\longrightarrow (d)$

(Proof of the claim :) Let $x \geq 1$

$$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x} < \sum_{n=1}^{\infty} e^{-n\pi x}$$

$$= e^{-\pi x}\left(1 + e^{-\pi x} + e^{-2\pi x} + \cdots\right)$$

Now $x \geq 1$, $-\pi x \leq -\pi$

$$\Rightarrow e^{-\pi x} \leq e^{-\pi} < 1/2$$

$$< e^{-\pi x}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$$

$$= 2e^{-\pi x}.$$

$$\Rightarrow \omega(x) = O\left(e^{-\pi x}\right) \text{ as } x \to \infty,$$

From ⓒ & ⓓ, for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)} + \int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)dx\right\}$$

$$= \underbrace{\frac{\pi^{s/2}}{2\Gamma\left(\frac{s}{2}+1\right)(s-1)}}_{\substack{\text{analytic except} \\ \text{for a simple pole} \\ \text{at } s=1,}} + \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}\right)}\int_1^\infty \omega(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right)dx \quad \text{ⓓ}$$

(entire fn. of s)

This gives analytic continuation of $\zeta(s)$ in $\mathbb{C} \setminus \{1\}$.

Also from (c) RHS is unchanged if we replace s by $1-s$.

This proves the functional eqn.

Remark: (1) From (d), we readily see that $\zeta(0) = -\frac{1}{2}$.

(2) $\zeta(-1) = -1/12$ (check using funct. eqn.)