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MA 631 – SPECIAL FUNCTIONS – Lec. 13

Euler's integral representation for ${}_2F_1(a,b;c;z)$

Thm. 4.1 For $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ & $|\arg(1-z)| < \pi$,

$$ {}_2F_1\left(\begin{matrix} a \;\; b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-1}(1-tz)^{-a}\,dt $$

Proof: Let $|z| < 1$ and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$.
Then by defn.,

$$ {}_2F_1\left(\begin{matrix} a \;\; b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n\, n!} $$

$$ = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \sum_{n=0}^{\infty} \left(\frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}\right) \frac{(a)_n z^n}{n!} \qquad \text{I} $$

$$ = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt \;\frac{(a)_n z^n}{n!} $$

We want to interchange the order of summation & integration. For that we need to show the following:

(i) $\sum_{n=0}^{\infty} U_n(t)$ converges uniformly on $(0,1)$,

where $U_n(t) := \dfrac{(a)_n (tz)^n}{n!}$

(ii) $$\sum_{n=0}^{\infty}\int_0^1 \left| t^{b+n-1}(1-t)^{c-b-1} U_n(t) \right| dt$$

or $$\int_0^1 \sum_{n=0}^{\infty} \left| t^{b+n-1}(1-t)^{c-b-1} U_n(t) \right| dt$$

converge.

To that end, note that (i) is true, for,

$$|U_n(t)| = \left| \frac{(a)_n (tz)^n}{n!} \right|$$

$$= \left| \frac{a(a+1)\cdots(a+n-1)}{n!} (tz)^n \right|$$

$$\leq \frac{|a|(|a|+1)\cdots(|a|+(n-1))}{n!} (t|z|)^n$$

$$\leq \frac{(|a|)_n |z|^n}{n!}$$

& $$\sum_{n=0}^{\infty} \frac{(|a|)_n |z|^n}{n!} = (1-|z|)^{-|a|} \quad (\because |z|<1)$$

Hence by Weierstrass-M test, $\sum_{n=0}^{\infty} U_n(t)$ conv. unif. in t on $(0,1)$ & hence (i) is satisfied,

(ii) $$\sum_{n=0}^{\infty} \int_0^1 \left| t^{b+n-1} (1-t)^{c-b-1} U_n(t) \right| dt$$

$$\leq \sum_{n=0}^{\infty} \int_0^1 t^{Re(b)+n-1} (1-t)^{Re(c-b)-1} \frac{(|a|)_n |z|^n}{n!} dt$$

$$= \sum_{n=0}^{\infty} B\left(Re(b)+n, Re(c-b) \right) \frac{(|a|)_n |z|^n}{n!}$$

$$= \Gamma(Re(c-b)) \sum_{n=0}^{\infty} \frac{(|a|)_n \Gamma(Re(b)+n)}{\Gamma(Re(c)+n)} \frac{|z|^n}{n!}$$

$$= \frac{\Gamma(Re(c-b))\Gamma(Re(c))}{\Gamma(Re(b))} \sum_{n=0}^{\infty} \frac{(|a|)_n (Re(b))_n}{(Re(c))_n} \frac{|z|^n}{n!}$$

$$< \infty. \quad (\because |z|<1)$$

Hence the interchange is justified.

From (I),

$${}_2F_1\left({a \quad b \atop c} ; z \right)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \sum_{n=0}^{\infty} t^{b+n-1} (1-t)^{c-b-1} \frac{(a)_n z^n}{n!} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(\sum_{n=0}^{\infty} \frac{(a)_n (tz)^n}{n!}\right) dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt,$$

This proves the result for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ & $|z|<1$.

For analytic continuation, note first of all that the integral

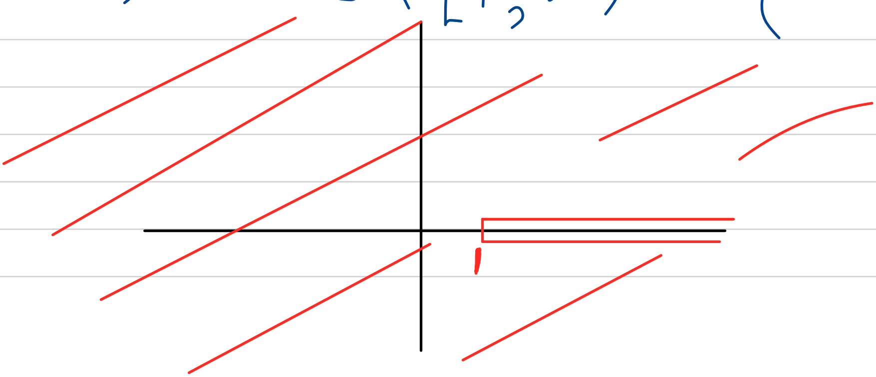
$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt$$

is valid as long as we omit the branch cut of $(1-tz)^{-a} = e^{-a\log(1-tz)}$,

i.e. $z \in \mathbb{C} \setminus \{w : 1 - tw \le 0\}$

$\Rightarrow z \in \mathbb{C} \setminus \{w : w \geq \frac{1}{t}\}$

$\Rightarrow z \in \mathbb{C} \setminus [1, \infty)$ $\quad (\because t \in [0,1])$,



Also, by one of the earlier theorems (Thm. ⊛ of Lec. 6), we have that
$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$ is analytic as a function of z in $\mathbb{C}\setminus[1,\infty)$.

Hence we can analytically continue ${}_2F_1\left(\begin{matrix}a\ \ b\\ c\end{matrix};z\right)$ outside of the unit disk by means of the integral
$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.$$

Note that if $z\in[1,\infty)$,
then $1-z\in(-\infty,0]$
$\Rightarrow$ $-\pi<\arg(1-z)<\pi$
or, in other words,
$$|\arg(1-z)|<\pi.$$ ∎

Transformations of ${}_2F_1(a,b;c;z)$:

① ${}_2F_1\left(\begin{matrix}a\ \ b\\ c\end{matrix};z\right)=(1-z)^{-a}\,{}_2F_1\left(\begin{matrix}a\ \ c-b\\ c\end{matrix};\frac{z}{z-1}\right)$
(Pfaff)
valid for $\mathrm{Re}(c)>\mathrm{Re}(b)>0$
& $|\arg(1-z)|<\pi$

Proof: ${}_2F_1(a, b; c; z)$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

Let $t = 1-u$, $dt = -du$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-u)^{b-1} u^{c-b-1} (1-(1-u)z)^{-a}\,du$$

$(1-z+uz)^{-a}$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} (1-z)^{-a} \int_0^1 u^{c-b-1}(1-u)^{b-1}\left(1+\frac{uz}{1-z}\right)^{-a} du$$

$$= (1-z)^{-a} \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 u^{(c-b)-1}(1-u)^{b-1}\left(1-\frac{uz}{z-1}\right)^{-a} dt$$

$$= (1-z)^{-a}\, {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{z}{z-1}\right)$$ (using Thm. 4.1)

$\boxtimes$