7/11/22

MA 631 – SPECIAL FUNCTIONS – Lec. 15

Remark: The above argument for the convergence of ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right)$ works exactly the same for ${}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}; z\right)$, where $|z| = 1$.

Gauss' representation for ${}_2F_1(a, b; c; 1)$

Thm. 4.2

$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, $$

whenever $Re(c-a-b) > 0$, $Re(c) > Re(a) > 0$
$Re(c) > Re(b) > 0$

Proof: From last lecture, we see that ${}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}; 1\right)$ converges when $Re(c-a-b) > 0$.

By Thm. 4.1, for $Re(c) > Re(b) > 0$,

$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt $$

Let $z \to 1$ so that

$$ \lim_{z\to 1} {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \lim_{z\to 1} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt $$

$$ \Rightarrow {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,dt $$

$$= \frac{\Gamma(c)}{\cancel{\Gamma(b)}\,\Gamma(c-b)} \cdot \frac{\cancel{\Gamma(b)}\,\Gamma(c-a-b)}{\Gamma(c-a)}$$

$$= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

Hypergeometric differential equation

Let θ be the differential operator defined by

$$\theta = z\frac{d}{dz}.$$

For eg., $\theta z^{\mu} = z\frac{d}{dz} z^{\mu} = z(\mu z^{\mu-1}) = \mu z^{\mu}$.

Also, $\theta(\theta+c-1)\,z^n$

$$= \theta\left[\theta(z^n) + (c-1)z^n\right]$$

$$= \theta\left[n z^n + (c-1)z^n\right]$$

$$= (n+c-1)\,\theta(z^n)$$

$$= n(n+c-1)\,z^n$$

$$\Rightarrow \theta(\theta+c-1)\,{}_2F_1(a,b;c;z)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!}\, \theta(\theta+c-1)(z^n)$$

$$= \sum_{n=0}^{\infty} \frac{n(n+c-1)(a)_n (b)_n}{(c)_n\, n!}\, z^n$$

$$= \sum_{n=1}^{\infty} \frac{n(n+c-1)(a)_n(b)_n}{(c)_n\, n!} z^n$$

$$= \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(n-1)!} (n+c-1) z^n$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}\, n!} (n+c)\, z^{n+1}$$

$$= z \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!} \left((a+n)(b+n) z^n\right)$$

Now $(\theta+a)(\theta+b)(z^n)$

$$= (\theta+a)\left[n z^n + b z^n\right]$$

$$= (n+b)(\theta+a)(z^n)$$

$$= (n+b)(n+a) z^n$$

$$\Rightarrow \theta(\theta+c-1)\, {}_2F_1(a,b;c;z)$$

$$= z \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!} (\theta+a)(\theta+b)(z^n)$$

$$= z(\theta+a)(\theta+b)\, {}_2F_1(a,b;c;z)$$

$$\Rightarrow \theta(\theta+c-1)\, {}_2F_1(a,b;c;z) = z(\theta+a)(\theta+b)\, {}_2F_1(a,b;c;z).$$