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MA 631 - Special Functions - Lec. 5

Euler's summation formula

Thm. 5 Let the function $f:[0,1]\to\mathbb{C}$ have k continuous derivatives for $k\in\mathbb{N}\cup\{0\}$. Then for $k\geq 1$, we have

$$f(1)=\int_0^1 f(x)\,dx+\sum_{i=1}^{k}\frac{(-1)^i B_i}{i!}\left(f^{(i-1)}(1)-f^{(i-1)}(0)\right)+R_k,$$

where

$$R_k=\frac{(-1)^{k+1}}{k!}\int_0^1 f^{(k)}(x)\,B_k(x)\,dx.$$

Proof: (By induction on k)

(i) when $k=1$, we have to show

$$f(1)=\int_0^1 f(x)\,dx+-B_1\left(f(1)-f(0)\right)+\int_0^1 f'(x)\,B_1(x)\,dx,$$

or in other words:

$$\int_0^1 f'(x)B_1(x)\,dx=-\int_0^1 f(x)\,dx+\frac{1}{2}\left(f(1)+f(0)\right),$$

$$\left(\because B_1=-\tfrac{1}{2}\right).$$

Performing integration by parts, we have

$$\int_0^1 f'(x) B_1(x)\, dx$$

$$= \Big[B_1(x) f(x)\Big]_0^1 - \int_0^1 B_1'(x) f(x)\, dx$$

$$= B_1(1) f(1) - B_1(0) f(0) - \int_0^1 (x - 1/2)' f(x)\, dx$$

$$= \left(x - \frac{1}{2}\right)\Big|_{x=1} \cdot f(1) - \left(-\frac{1}{2}\right) f(0) - \int_0^1 f(x)\, dx$$

$$= \frac{1}{2}\big(f(1) + f(0)\big) - \int_0^1 f(x)\, dx.$$

(ii) Assume it is true for $k = m$, say.

(iii) Show that the following is true for $k = m+1$:

$$f(1) = \int_0^1 f(x) + \sum_{i=1}^{m+1} \frac{(-1)^i B_i}{i!} \left(f^{(i-1)}(1) - f^{(i-1)}(0) \right)$$

$$+ R_{m+1},$$

where $R_{m+1} = \dfrac{(-1)^{m+2}}{(m+1)!} \displaystyle\int_0^1 f^{(m+1)}(x)\, B_{m+1}(x)\, dx.$

$$= f(1) - R_m + \frac{(-1)^{m+1} B_{m+1}}{(m+1)!}\left(f^{(m)}(1) - f^{(m)}(0)\right)$$

$$+ R_{m+1},$$

Hence we have to show

$$R_{m+1} - R_m = \frac{(-1)^m B_{m+1}}{(m+1)!}\left(f^{(m)}(1) - f^{(m)}(0)\right).$$

$$LHS = \frac{(-1)^{m+2}}{(m+1)!}\int_0^1 f^{(m+1)}(x) B_{m+1}(x)\,dx - \frac{(-1)^{m+1}}{m!}\int_0^1 f^{(m)}(x) B_m(x)\,dx$$

$$= \frac{(-1)^{m+2}}{(m+1)!}\left\{\left[B_{m+1}(x) f^{(m)}(x)\right]_0^1 - \int_0^1 B_{m+1}'(x) f^{(m)}(x)\,dx\right\}$$

$$- \frac{(-1)^{m+1}}{m!}\int_0^1 f^{(m)}(x) B_m(x)\,dx$$

$$= \frac{(-1)^m}{(m+1)!}\left\{\left(B_{m+1}(1) f^{(m)}(1) - B_{m+1} f^{(m)}(0)\right)\right.$$

$$\left. - (m+1)\int_0^1 \cancel{B_m(x) f^{(m)}(x)\,dx}\right\}$$

$$- \frac{(-1)^{m+1}}{m!}\int_0^1 \cancel{f^{(m)}(x) B_m(x)\,dx}$$

$$= \frac{(-1)^m}{(m+1)!} B_{m+1} \left(f^{(m)}(1) - f^{(m)}(0) \right)$$

$$\left(\because B_{m+1}(1) = B_{m+1}(0) \text{ for } m > 0 \right).$$

This proves the required result.

- The above formula can be generalized to the interval $[j-1, j]$:

$$f(j) = \int_{j-1}^{j} f(x)\,dx + \sum_{i=1}^{k} \frac{(-1)^i B_i}{i!} \left(f^{(i-1)}(j) - f^{(i-1)}(j-1) \right) + R_k, \longrightarrow (A)$$

where

$$R_k = \frac{(-1)^{k+1}}{k!} \int_{j-1}^{j} f^{(k)}(x)\, \tilde{B}_k(x)\, dx,$$

where

$$\tilde{B}_k(x) = B_k(x), \quad 0 \le x < 1,$$
$$\tilde{B}_k(x+1) = \tilde{B}_k(x), \quad \forall x \in \mathbb{R}. \longrightarrow (B)$$

Summing (A) over $1 \le j \le n$ gives

$$\boxed{\sum_{j=1}^{n} f(j) = \int_0^n f(x)\,dx + \sum_{i=1}^{k} \frac{(-1)^i B_i}{i!} \left(f^{(i-1)}(n) - f^{(i-1)}(0) \right) + \frac{(-1)^{k+1}}{k!} \int_0^n f^{(k)}(x)\, \tilde{B}_k(x)\, dx \quad \text{(using (B))}.}$$

Special case when $k=1$:

$$\sum_{j=1}^{n} f(j) = \int_0^n f(x)\,dx + \frac{1}{2}\left(f(n)-f(0)\right) \qquad (*)$$

$$+ \int_0^n f'(x)\,\tilde{B}_1(x)\,dx.$$

Existence of Euler's constant :

$$\gamma := \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$

$$\approx 0.577\ldots$$

Let $f(x) = \dfrac{1}{x+1}$.

Replace n by $n-1$ in $(*)$ so that

$$\sum_{j=1}^{n-1}\frac{1}{j+1} = \int_0^{n-1}\frac{1}{x+1}\,dx + \frac{1}{2}\left(\frac{1}{n}-1\right)$$

$$- \int_0^{n-1}\frac{\tilde{B}_1(x)}{(x+1)^2}\,dx$$

$$\Rightarrow 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$= \left[\log(x+1)\right]_0^{n-1} + \frac{1}{2n} + \frac{1}{2} - \int_0^{n-1}\frac{\tilde{B}_1(x)\,dx}{(1+x)^2}$$

$$= \log n + \frac{1}{2n} + \frac{1}{2} - \int_0^{n-1} \frac{\tilde{B}_1(x)\,dx}{(1+x)^2}$$

$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$$

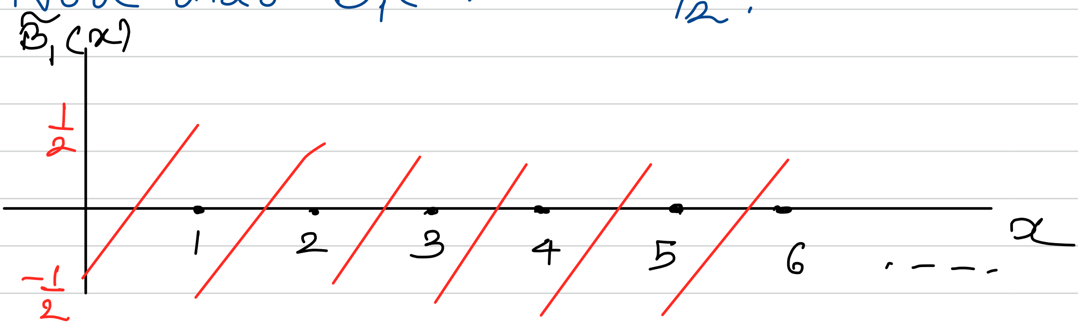
$$= \frac{1}{2n} + \frac{1}{2} - \int_0^{n-1} \frac{\tilde{B}_1(x)\,dx}{(1+x)^2} \quad \text{——} (C)$$

Now take $\lim_{n\to\infty}$ on RHS:

$$\lim_{n\to\infty} \text{RHS} = \frac{1}{2} - \int_0^{\infty} \frac{\tilde{B}_1(x)\,dx}{(1+x)^2}, \text{ provided}$$

$\int_0^{\infty} \frac{\tilde{B}_1(x)\,dx}{(1+x)^2}$ converges.

Note that $B_1(x) = x - 1/2$.



$$\Rightarrow \quad |\tilde{B}_1(x)| \le 1/2$$

$$\Rightarrow \int_0^\infty \frac{\tilde{B}_1(x)\,dx}{(1+x)^2} \le \frac{1}{2}\int_0^\infty \frac{dx}{(1+x)^2} = \left[-\frac{1}{2(1+x)}\right]_0^\infty$$

$$= 0 + \frac{1}{2} = \frac{1}{2},$$

$$\Rightarrow \int_0^\infty \frac{\tilde{B}_1(x)\,dx}{(1+x)^2} \text{ converges.}$$

From (C), we deduce ~~that~~

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n\right) \text{ exists.}$$