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MA 631 - Special Functions - Lec. 6

Gamma function is defined for $Re(z)>0$ by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt .$$

Lemma 6 Let $S=\{z : a\le Re(z)\le A\}$, where $0<a<A<\infty$.

(a) For every $\varepsilon>0$, $\exists\ \delta>0 \ni \forall z\in S$,

$$\left|\int_\alpha^\beta e^{-t} t^{z-1}\, dt\right| < \varepsilon ,$$

whenever $0<\alpha<\beta<\delta$.

(b) For every $\varepsilon>0$, $\exists$ a $K>0 \ni \forall z\in S$,

$$\left|\int_\alpha^\beta e^{-t} t^{z-1}\, dt\right| < \varepsilon ,$$

whenever $\beta>\alpha>K$

Proof: Let $0<t\le 1$. For $z\in S$, we have

$$t^{Re(z)-1} = |t^{z-1}| \le |t^{a-1}| = t^{a-1}$$

$$\Rightarrow |e^{-t} t^{z-1}| \le t^{Re(z)-1} \le t^{a-1}.$$

$$(\because e^{-t}\le 1 ,\ \text{since } 0<t\le 1)$$

For $0 < \alpha < \beta < 1$,

$$\Rightarrow \left| \int_\alpha^\beta e^{-t} t^{z-1}\, dt \right| \le \int_\alpha^\beta \left| e^{-t} t^{z-1} \right| dt$$

$$\le \int_\alpha^\beta t^{a-1}\, dt = \left[\frac{t^a}{a} \right]_\alpha^\beta = \frac{\beta^a - \alpha^a}{a}$$

If $\varepsilon > 0$, choose δ, $0 < \delta < 1$, s.t.

$$\frac{\beta^a - \alpha^a}{a} < \varepsilon \text{ for } |\alpha - \beta| < \delta$$

$$\Rightarrow \left| \int_\alpha^\beta e^{-t} t^{z-1}\, dt \right| \le \varepsilon \text{ for } 0 < \alpha < \beta < \delta.$$

(b) $\forall z \in S$, we have $|t^{z-1}| < t^{A-1}$ for $t \geq 1$,

Note that $t^{A-1} e^{-t/2}$ is continuous on $[1, \infty)$

Also $\lim_{t \to \infty} t^{A-1} e^{-t/2} = 0$.

Hence $\exists$ a constant $c \ni$

$$\left| t^{A-1} e^{-t/2} \right| \le c \quad \forall t \geq 1.$$

Hence $\left| e^{-t} t^{z-1} \right| \le c e^{-t/2}$

$\forall z \in S$ & $t \geq 1$,

If $\beta > \alpha > 1$, then

$$\left| \int_\alpha^\beta e^{-t} t^{z-1}\, dt \right| \le \int_\alpha^\beta c e^{-t/2}\, dt$$

$$= c\left[\frac{e^{-t/2}}{-1/2}\right]_{\alpha}^{\beta} = -2c\left(e^{-\beta/2} - e^{-\alpha/2}\right).$$

Now use continuity of $-2c\,e^{-t}$ on $[1,\infty)$ to see that, given $\varepsilon > 0$, $\exists\, K > 1 \ni \forall\, \beta > \alpha > K$,

$$-2c\left(e^{-\beta/2} - e^{-\alpha/2}\right) < \varepsilon .$$

$$\Rightarrow \quad \left|\int_{\alpha}^{\beta} e^{-t}\, t^{z-1}\, dt\right| < \varepsilon, \text{ for } \beta > \alpha > K.$$

$\boxtimes$

Proving that the integral $\int_0^{\infty} e^{-t}\, t^{z-1}\, dt$, $Re(z) > 0$, is uniformly convergent on compact subsets of $Re(z) > 0$:

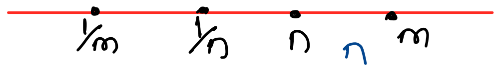
$$\int_0^{\infty} e^{-t}\, t^{z-1}\, dt = \int_0^{m} e^{-t}\, t^{z-1}\, dt + \underbrace{\int_m^{M} e^{-t}\, t^{z-1}\, dt}_{\text{bdd.}} + \int_M^{\infty} e^{-t}\, t^{z-1}\, dt$$

bdd. (cont. fn. on a compact set $[m, M]$)

$$f_n(z) := \int_{1/n}^{n} e^{-t}\, t^{z-1}\, dt$$

Show that $\{f_n(z)\}_{n=1}^{\infty}$ is Cauchy.

$$m > n \implies \frac{1}{m} < \frac{1}{n}$$



$$|f_m(z) - f_n(z)| = \left| \int_{1/m}^{m} e^{-t} t^{z-1}\,dt - \int_{1/n}^{n} e^{-t} t^{z-1}\,dt \right|$$

$$= \left| \int_{1/m}^{1/n} e^{-t} t^{z-1}\,dt - \int_{n}^{m} e^{-t} t^{z-1}\,dt \right|$$

$$\leq \varepsilon.$$

Use the fact that the space of all analytic functions on $Re(z) > 0$ is complete.

$\implies \{f_n(z)\}_{n=1}^{\infty}$ converges & it converges to $\int_0^\infty e^{-t} t^{z-1}\,dt$ converges.

This also tells us that $\int_0^\infty e^{-t} t^{z-1}\,dt$ is continuous fn. of z.

Thm. * Let t be a real variable ranging over finite or infinite interval (a, b), and z be a complex variable over a domain Ω. Assume that the function $f : (\Omega \times (a,b)) \to \mathbb{C}$ satisfies the following conditions:

① f is a continuous function of both the variables.
② For each fixed value of t, $f(\cdot, t)$ is a holomorphic

function of the first variable.

(3) The integral $F(z) = \int_a^b f(z,t)\,dt$, $z \in \Omega$ converges uniformly at both the limits in any compact set in Ω.

Then $F(z)$ is holomorphic in Ω, and its derivatives of all orders may be found by differentiating under the sign of integration.

Let $\Omega = \{z : \operatorname{Re}(z) > 0\}$, $a = 0$, $b = \infty$,

$$f(z,t) = e^{-t}\, t^{z-1}$$

Using Lemma 6 & Thm. *, we have now proved the following theorem:

Thm. 7 $\Gamma(z)$ is analytic in $\operatorname{Re}(z) > 0$.