18/11/22

MA 631 – Special Functions – Lec. 7

For $\operatorname{Re}(z) > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt.$$

Thm. 7 $\Gamma(z)$ is analytic in $\operatorname{Re}(z) > 0$.

(Proved last time)

Functional equation of $\Gamma(z)$

Thm. 8 $\Gamma(z+1) = z\Gamma(z)$

(Cor. 9) Let $z \in \mathbb{N}$, say $z = n$. Then

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots \\
&= n(n-1)(n-2)\cdots 2 \cdot \Gamma(2) \\
&= n(n-1)\cdots 2 \cdot 1 \cdot \Gamma(1) \\
&= n!
\end{aligned}$$

($\because \Gamma(1) = 1$, as can be seen from the defn.)

Proof of Thm. 8: For $\operatorname{Re}(z) > -1$,

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z} \, dt.$$

Now consider $\operatorname{Re}(z) > 0$. Performing integration by parts, we get

$$\Gamma(z+1) = \left[-t^{z} e^{-t}\right]_0^\infty + z\int_0^\infty t^{z-1} e^{-t}\, dt \quad \text{——— (1)}$$

Now $\lim_{t\to\infty} t^z e^{-t} = 0$

Also $\lim_{t\to 0} t^z e^{-t} = 0$ (since $Re(z) > 0$). ——— (2)

$\left(\because \lim_{t\to 0} |t^z e^{-t}| = \lim_{t\to 0} t^{Re(z)} e^{-t} = 0.\right)$

From (1) & (2), we see that

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for } Re(z) > 0.$$

$\boxtimes$

Meromorphically continuing $\Gamma(z)$ in $Re(z) \leq 0$

We have shown that for $Re(z) > 0$,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

Observe that RHS is analytic in $Re(z) > -1$, $z \neq 0$ (using Thm. 7.)

Thus, by analytic continuation, we see that $\Gamma(z)$ is analytic in $Re(z) > -1$, $z \neq 0$, upon defining it to be $\frac{\Gamma(z+1)}{z}$ in this region.

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \quad \text{in } \operatorname{Re}(z) > -1,\ z \neq 0$$

$$= \frac{\Gamma(z+2)}{z(z+1)} \quad \text{(replacing } z \text{ by } z+1 \text{ in Thm. 7)}$$

Repeat this procedure step by step to enter into the left-half plane.

Prym's decomposition of $\Gamma(z)$

Let $\operatorname{Re}(z) > 0$. Then

$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1}\, dt + \int_1^\infty e^{-t} t^{z-1}\, dt .$$

Observe that $\int_1^\infty e^{-t} t^{z-1}\, dt$ is an entire function of z.

Consider $\int_0^1 e^{-t} t^{z-1}\, dt = \int_0^1 t^{z-1} \sum_{n=0}^\infty \frac{(-t)^n}{n!}\, dt$

Since $\sum_{n=0}^\infty \frac{(-t)^n}{n!}$ converges uniformly on $(0,1)$, we can interchange the order of summation and integration. Hence

$$\Gamma(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{z+n-1}\, dt + \int_1^\infty e^{-t} t^{z-1}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{t^{z+n}}{z+n}\right]_0^1 + \int_1^{\infty} e^{-t} t^{z-1} \, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1} \, dt \, , \quad z \neq 0, -1, -2, \ldots$$

$\Rightarrow \Gamma(z)$ has simple poles at $z = 0, -1, -2, -3, \ldots$

Consider the simple pole at $z = -m$. Then

$$\lim_{z \to -m} (z+m)\Gamma(z)$$

$$= \lim_{z \to -m} (z+m) \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1} \, dt \right\}$$

$$= \lim_{z \to -m} (z+m) \left\{ \frac{(-1)^m}{m!(z+m)} + \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1} \, dt \right\}$$

$$= \frac{(-1)^m}{m!} .$$

Thus Gamma function has simple poles at $z = -n$ $(n \in \mathbb{N} \cup \{0\})$, with residue $\frac{(-1)^n}{n!}$.

Beta function

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \quad (\mathrm{Re}(p)>0, \ \mathrm{Re}(q)>0).$$

Clearly, $B(p,q) = B(q,p)$

Thm. 10 $\quad B(p,q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$

Proof: Consider

$$I(p,q) := \int_0^\infty\int_0^\infty x^{2p-1}\,y^{2q-1}\,e^{-(x^2+y^2)}\,dx\,dy$$

$$= \left(\int_0^\infty e^{-x^2}x^{2p-1}\,dx\right)\left(\int_0^\infty e^{-y^2}y^{2q-1}\,dy\right)$$

Now for $\mathrm{Re}(p)>0$,

$$\Gamma(p) = \int_0^\infty e^{-t}\,t^{p-1}\,dt$$

$$\overset{(t=x^2)}{=} \int_0^\infty e^{-x^2}(x^2)^{p-1}(2x\,dx)$$

$$= 2\int_0^\infty e^{-x^2}x^{2p-1}\,dx$$

Hence, $I(p,q) = \frac{1}{4}\Gamma(p)\Gamma(q)$ ——— (1)

Now let us evaluate $I(p,q)$ by using polar coordinates. So let

$$x = r\cos\theta, \quad y = r\sin\theta$$

Then $dx\,dy \overset{?}{=} r\,dr\,d\theta$.

$$\Rightarrow I(p,q) = \int_0^\infty \int_0^{\pi/2} (r\cos\theta)^{2p-1} (r\sin\theta)^{2q-1} e^{-r^2} r\,dr\,d\theta$$

$$= \left(\int_0^\infty e^{-r^2} r^{2(p+q)-1}\,dr\right)\left(\int_0^{\pi/2} (\cos\theta)^{2p-1} (\sin\theta)^{2q-1}\,d\theta\right)$$

$$= \frac{1}{2}\Gamma(p+q)\cdot\left(\int_0^{\pi/2} (\cos\theta)^{2p-1} (\sin\theta)^{2q-1}\,d\theta\right).$$

But $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \quad \left(\begin{matrix} Re(p)>0 \\ Re(q)>0 \end{matrix}\right)$

Let $t = \sin^2\theta \Rightarrow dt = 2\sin\theta\cos\theta\,d\theta$

$$\Rightarrow B(p,q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1} (\cos^2\theta)^{q-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}\,d\theta$$

$$\Rightarrow \int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}\,d\theta = \frac{1}{2}B(p,q).$$

$$\Rightarrow I(p,q) = \frac{1}{4}\Gamma(p+q)\,B(p,q) \quad \text{———} \quad 2$$

From 1 & 2,

$$\frac{1}{4}\Gamma(p)\Gamma(q) = \frac{1}{4}\Gamma(p+q)\,B(p,q)$$

$$\Rightarrow B(q,p) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

$$\Rightarrow B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

$\blacksquare$