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MA 631 - Special Functions - Tut. 1

1. $B_n(-x) = (-1)^n \left(B_n(x) + n x^{n-1}\right)$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(B_n(x) + n x^{n-1}\right)}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{B_n(x)(-z)^n}{n!} + \sum_{n=1}^{\infty} \frac{x^{n-1}(-z)^n}{(n-1)!}$$

$$= \frac{(-z)\, e^{-xz}}{e^{-z}-1} - z \sum_{n=0}^{\infty} \frac{(-xz)^n}{n!}$$

$$= \frac{-z\, e^{-xz}}{e^{-z}-1} - z e^{-xz}$$

$$= -z e^{-xz} \left(\frac{1}{e^{-z}-1} + 1 \right)$$

$$= -z e^{-xz} \frac{e^{-z}}{e^{-z}-1}$$

$$= \frac{z e^{-xz}}{e^{z}-1} = \sum_{n=0}^{\infty} \frac{B_n(-x) z^n}{n!}$$

$$\Rightarrow B_n(-x) = (-1)^n \left(B_n(x) + n x^{n-1}\right).$$

② (Raabe)

$$\frac{1}{m}\sum_{k=0}^{m-1} B_n\left(x+\frac{k}{m}\right) = m^{-n} B_n(mx).$$

Proof:

$$\sum_{n=0}^{\infty}\left(\frac{1}{m}\sum_{k=0}^{m-1} B_n\left(x+\frac{k}{m}\right)\right)\frac{z^n}{n!}$$

$$= \frac{1}{m}\sum_{k=0}^{m-1}\sum_{n=0}^{\infty} B_n\left(x+\frac{k}{m}\right)\frac{z^n}{n!}$$

$$= \frac{1}{m}\sum_{k=0}^{m-1} \frac{z\, e^{(x+k/m)z}}{e^z-1}$$

$$= \frac{1}{m}\,\frac{z e^{xz}}{e^z-1}\sum_{k=0}^{m-1}\left(e^{z/m}\right)^k$$

$$= \frac{1}{m}\,\frac{z e^{xz}}{e^z-1}\cdot\frac{1-\left(e^{z/m}\right)^m}{1-e^{z/m}}$$

$$= \frac{-1}{m}\,\frac{z e^{xz}}{1-e^{z/m}} = \frac{z/m\; e^{mx(z/m)}}{e^{z/m}-1}$$

$$= \sum_{n=0}^{\infty} B_n(mx)\,\frac{(z/m)^n}{n!}.$$

$$= \sum_{n=0}^{\infty} m^{-n} B_n(mx) \frac{z^n}{n!}.$$

Now compare coefficients on both sides.

$\boxtimes$

③ (i) Prove $\frac{z}{2}\left(\coth\left(\frac{z}{2}\right) - 1\right) = \frac{z}{e^z - 1}$.

Note that $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$.

$$= \frac{\left(\frac{e^x + e^{-x}}{2}\right)}{\left(\frac{e^x - e^{-x}}{2}\right)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$= \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{(e^{2x} - 1) + 2}{e^{2x} - 1}$$

$$= 1 + \frac{2}{e^{2x} - 1}.$$

$$\Rightarrow \frac{z}{2}\left(\coth\left(\frac{z}{2}\right) - 1\right)$$

$$= \frac{z}{2} \cdot \frac{2}{e^z - 1} = \frac{z}{e^z - 1}.$$

(ii) $2\coth(2z) - \coth(z) = \tanh(z)$

$$\text{LHS} = 2\left(\frac{e^{4x}+1}{e^{4x}-1}\right) - \left(\frac{e^{2x}+1}{e^{2x}-1}\right)$$

$$= \frac{1}{e^{2x}-1}\left\{\frac{2(e^{4x}+1)}{e^{2x}+1} - (e^{2x}+1)\right\}$$

$$= \frac{1}{e^{2x}-1}\left\{2(e^{4x}+1) - (e^{2x}+1)^2\right\}$$

$$= \frac{1}{e^{2x}-1}\left\{\frac{2e^{4x}+2-e^{4x}-2e^{2x}-1}{(e^{2x}+1)}\right\}$$

$$= \frac{1}{e^{2x}-1}\left\{\frac{e^{4x}-2e^{2x}+1}{(e^{2x}+1)}\right\}$$

$$= \frac{e^{2x}-1}{e^{2x}+1} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \tanh x,$$

Goal: $\tanh(z) = 1 - \sum_{n=0}^{\infty} \frac{T_n z^n}{n!} \quad \left(|z| < \frac{\pi}{2}\right)$

where T_n is defined by

$$B_n = \frac{-nT_{n-1}}{2^n(2^n-1)}, \quad n \in \mathbb{N},$$

To that end first note that

$$\frac{z}{e^{z}-1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}$$

Now $\tanh z = 2\coth(2z) - \coth(z)$

$$= \frac{2\left(\frac{4z}{2}\left(\coth\left(\frac{4z}{2}\right)-1\right)\right)+4z}{2z} - \left\{\frac{\left(\frac{2z}{2}\left(\coth\left(\frac{2z}{2}\right)-1\right)\right)+z}{z}\right\}$$

$$= \frac{2\cdot\frac{4z}{e^{4z}-1}+4z}{2z} - \frac{\frac{2z}{e^{2z}-1}+z}{z}$$

$$= 1+\frac{1}{z}\left(\frac{4z}{e^{4z}-1}-\frac{2z}{e^{2z}-1}\right)$$

$$= 1+\frac{1}{z}\left\{\sum_{n=0}^{\infty}\frac{B_n(4z)^n}{n!}-\sum_{n=0}^{\infty}\frac{B_n(2z)^n}{n!}\right\}$$

$$= 1+\sum_{n=0}^{\infty}\frac{B_n z^{n-1}}{n!}\left(4^n-2^n\right)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{2^n(2^n-1) B_n}{n!} z^{n-1}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{T_{n-1} z^{n-1}}{(n-1)!} \qquad \left(B_n = \frac{-n T_{n-1}}{2^n(2^n-1)} \right)$$

$$= 1 - \sum_{n=0}^{\infty} \frac{T_n z^n}{n!}.$$

(iii) Prove $\tan z = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} T_{2n+1} z^{2n+1}}{(2n+1)!}$

Proof:

$$\tan z = \frac{\sin z}{\cos z} = \frac{-i \sinh(iz)}{\cosh(iz)}$$

$$= -i \tanh(iz)$$

$$= -i \left\{ 1 - \sum_{n=0}^{\infty} \frac{T_n (iz)^n}{n!} \right\}$$

$$= -i \left\{ 1 - \sum_{n=0}^{\infty} \frac{(-1)^n T_{2n} z^{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{i^{2n+1} T_{2n+1} z^{2n+1}}{(2n+1)!} \right\}$$

$$\left(B_n = \frac{-n T_{n-1}}{2^n(2^n-1)} \right)$$

$$= -i\left\{ 1 + \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}(2^{2n+1}-1)}{(2n+1)} B_{2n+1} \frac{z^{2n}}{(2n)!} \right.$$

$$\left. - i \sum_{n=0}^{\infty} (-1)^n \frac{T_{2n+1}\, z^{2n+1}}{(2n+1)!} \right\}$$

$$= -i\left\{ 1 + \frac{2(2-1)B_1}{1} - i \sum_{n=0}^{\infty} (-1)^n \frac{T_{2n+1}\, z^{2n+1}}{(2n+1)!} \right\}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{T_{2n+1}\, z^{2n+1}}{(2n+1)!}.$$