20/1/22

MA 631 – SPECIAL FUNCTIONS – Tut. 2

1 (a) $$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{2n}} = \frac{(-1)^{n+1}(2\pi)^{2n}(1-2^{1-2n})B_{2n}}{2(2n)!}$$

Observe that

$$B_{2n+1}(x) = 2(-1)^{n+1}(2n+1)! \sum_{k=1}^{\infty} \frac{\sin(2\pi k x)}{(2\pi k)^{2n+1}}$$

(ii) For $0 \le x \le 1$ and $n > 0$,

$$B_{2n}(x) = 2(-1)^{n+1}(2n)! \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{(2\pi k)^{2n}}$$

Let $x = 1/2$ in (ii) so that

$$B_{2n}(\tfrac{1}{2}) = 2(-1)^{n+1}(2n)! \sum_{m=1}^{\infty} \frac{(-1)^m}{(2\pi m)^{2n}}$$

Also $B_n(\frac{1}{2}) = (2^{1-n}-1)B_n$

$$\Rightarrow B_{2n}(\tfrac{1}{2}) = -(1-2^{1-2n})B_{2n}$$

Substitute these expressions & simplify.

(b) $$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}} = \frac{(-1)^{n+1}(2\pi)^{2n}(1-2^{-2n})B_{2n}}{2(2n)!}$$

Proof: $$\sum_{m=1}^{\infty} \frac{1}{m^{2n}} = \sum_{m=1}^{\infty} \frac{1}{(2m)^{2n}} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}}$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2n}} = (1-2^{-2n}) \sum_{m=1}^{\infty} \frac{1}{m^{2n}} \quad \text{--- } (*)$$

However, letting $x=0$ in (ii) gives

$$\sum_{m=1}^{\infty} \frac{1}{m^{2n}} = \frac{B_{2n}(2\pi)^{2n}}{2(-1)^{n+1}(2n)!} \quad \text{--- } (**)$$

Now combine $(*)$ & $(**)$ to complete the proof.

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(3) $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$

$$B(p,q) = \int_0^{\infty} \frac{x^{p-1}}{(1+x)^{p+q}}\,dx$$

1st proof: Let $x=\tan^2\theta$, $dx = 2\tan\theta\sec^2\theta\,d\theta$

when $x=0$, $\theta=0$; when $x=\infty$, $\theta=\pi/2$, Hence $\displaystyle\int_0^{\infty} \frac{x^{p-1}\,dx}{(1+x)^{p+q}}$

$$= \int_0^{\pi/2} \frac{(\tan^2\theta)^{p-1}}{(\sec^2\theta)^{p+q}} \cdot 2\tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{-\cancel{2p}+\cancel{2}+\cancel{2p}+2q-1\cancel{-2}} \, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1} \, d\theta$$

$$= B(p,q).$$

<u>2^{nd} proof:</u> $\displaystyle B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$

Let $\displaystyle x = \frac{t}{1+t}$ $\quad \displaystyle dx = \frac{(1+t)-t}{(1+t)^2}\,dt$

$$= \frac{dt}{(1+t)^2},$$

when $x=0$, $t=0$

when $x=1$, $t=\infty$.

$$B(p,q) = \int_0^\infty \left(\frac{t}{1+t}\right)^{p-1} \left(1 - \frac{t}{1+t}\right)^{q-1} \frac{dt}{(1+t)^2}$$

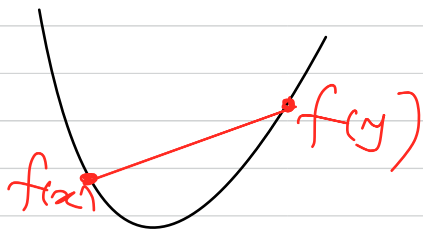
$$= \int_0^\infty \frac{t^{p-1}\,dt}{(1+t)^{p-1+q-1+2}} = \int_0^\infty \frac{t^{p-1}\,dt}{(1+t)^{p+q}},$$

5. $\Gamma(x)$ is log-convex on $(0,\infty)$.
i.e. $\log\Gamma(x)$ is convex on $(0,\infty)$

f is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for $0 \leq \lambda \leq 1$.



Hölder's inequality: if $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_a^b fg\,dx \leq \left(\int_a^b f^p\,dx\right)^{1/p} \left(\int_a^b g^q\,dx\right)^{1/q}.$$

To show:

$$\log\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \leq \frac{1}{p}\log\Gamma(x) + \frac{1}{q}\log\Gamma(y).$$

for $0 < x < \infty$,

$$\Gamma\left(\frac{x}{p}+\frac{y}{q}\right) \le (\Gamma(x))^{1/p} (\Gamma(y))^{1/q}.$$

Note that $\frac{x}{p}+\frac{y}{q} > 0$ so that

$$\Gamma\left(\frac{x}{p}+\frac{y}{q}\right) = \int_0^\infty e^{-t}\, t^{\frac{x}{p}+\frac{y}{q}-1}\, dt$$

$$= \int_0^\infty e^{-\left(\frac{1}{p}+\frac{1}{q}\right)t}\, t^{\frac{x}{p}+\frac{y}{q}-\left(\frac{1}{p}+\frac{1}{q}\right)}\, dt$$

$$= \int_0^\infty \left(e^{-\frac{t}{p}}\, t^{\frac{x-1}{p}}\right)\left(e^{-\frac{t}{q}}\, t^{\frac{y-1}{q}}\right) dt$$

Hölder's ineq.

$$\le \left(\int_0^\infty e^{-t}\, t^{x-1}\, dt\right)^{1/p} \left(\int_0^\infty e^{-t}\, t^{y-1}\, dt\right)^{1/q}$$

$$= (\Gamma(x))^{1/p} (\Gamma(y))^{1/q}.$$

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Bohr-Mollerup theorem.

If $f:(0,\infty)\to\mathbb{R}$ satisfies

1. $f(x) > 0$
2. $f(x+1) = x f(x)$
3. $\log f$ is convex on $(0,\infty)$,

then $f(x) = \Gamma(x)$.

(2) $$\int_0^{\pi} \frac{(\sin x)^{n-1}}{(a+b\cos x)^n}\,dx \qquad (a>b) \quad n\in\mathbb{N}$$

(4) $$B(x,y) = \frac{x+y}{y} B(x,y+1).$$

$$\iff \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x+y)}{y}\frac{\Gamma(x)\Gamma(y+1)}{\Gamma(x+y+1)}$$

Use $\Gamma(z+1) = z\Gamma(z)$.

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