MA 623: Homework 3 (Due February 26)

(Note: Justify all the relevant steps.)

1. Let $\omega(n)$ denote the number of prime divisors of the positive integer n. Find an asymptotic estimate for the sum $\displaystyle\sum_{n\le x} 2^{\omega(n)}$ with the error term $O\left(\sqrt{x}\log x\right)$.

2.Let $\varphi_1(n)$ be defined by

$$\varphi_1(n) := n\sum_{d|n}\frac{|\mu(d)|}{d}.$$

Prove that

$$\sum_{n\le x}\varphi_1(n) = \sum_{d\le\sqrt{x}}\mu(d)S\left(\frac{x}{d^2}\right),$$

where $S(x)=\sum_{k\le x}\sigma(k)$. Then deduce that for $x\ge 2$,

$$\sum_{n\le x}\varphi_1(n) = \frac{\zeta(2)}{2\zeta(4)}x^2 + O(x\log x).$$

3. Let $f(x)$ and $g(x)$ be positive, continuous functions on $[0,\infty)$, and set $F(x) = \int_0^x f(y)\,dy, G(x) = \int_0^x g(y)\,dy$. Show (by a counter-example) that the relation

$$f(x) = o(g(x)) \quad (x\to\infty)$$

does *not* imply

$$F(x) = o(G(x)) \quad (x\to\infty).$$

4. (a) Show that $\frac{1}{\phi} = \frac{1}{N} * f$, where $f = \frac{\mu^2}{N\cdot\phi}$.
   (b) Show that $\sum_{n=1}^{\infty} f(n) = O(1)$.
   (c) If $x\ge 2$, show that

$$\sum_{n\le x}\frac{1}{\phi(n)} = O(\log x).$$