MONOTONICITY RESULTS FOR DIRICHLET L-FUNCTIONS

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Abstract. We present some monotonicity results for Dirichlet L-functions associated to real primitive characters. We show in particular that these Dirichlet L-functions are far from being logarithmically completely monotonic. Also, we show that, unlike in the case of the Riemann zeta function, the problem of comparing the signs of $\frac{\partial^k}{\partial s^k} \log L(s, \chi)$ at any two points $s_1, s_2 > 1$ is more subtle.

1. Introduction

A function $f$ is said to be completely monotonic on $[0, \infty)$ if $f \in C'[0, \infty)$, $f \in C^\infty(0, \infty)$ and $(-1)^k f^{(k)}(t) \geq 0$ for $t > 0$ and $k = 0, 1, 2 \cdots$, i.e., the successive derivatives alternate in sign. The following theorem due to S.N. Bernstein and D. Widder gives a complete characterization of completely monotonic functions [10, p. 95]:

A function $f : [0, \infty) \to [0, \infty)$ is completely monotonic if and only if there exists a non-decreasing bounded function $\gamma$ such that $f(t) = \int_0^\infty e^{-st} d\gamma(s)$.

Lately, the class of completely monotonic functions have been greatly expanded to include several special functions, for example, functions associated to gamma and psi functions by Chen [9], Guo, Guo and Qi [15] and quotients of K-Bessel functions by Ismail [16]. A conjecture that certain quotients of Jacobi theta functions are completely monotonic was formulated by the first author and Solynin in [12], and slightly corrected later by the present authors in [13]. Certain other classes of such functions were introduced by Alzer and Berg [1], Qi and Chen [22]. Completely monotonic functions have applications in diverse fields such as probability theory [17], physics [4], potential theory [6], combinatorics [3] and numerical and asymptotic analysis [14], to name a few.

A close companion to the class of completely monotonic functions is the class of logarithmically completely monotonic functions. This was first studied, although implicitly, by Alzer and Berg [2]. A function $f : (0, \infty) \to (0, \infty)$ is said to be logarithmically completely monotonic [5] if it is $C^\infty$ and $(-1)^k [\log f(x)]^{(k)}(x) \geq 0$, for $k = 0, 1, 2, 3, \cdots$. Moreover, a function is said to be strictly logarithmically completely monotonic if $(-1)^k [\log f(x)]^{(k)}(x) > 0$. The following is true:
Every logarithmic completely monotonic function is completely monotonic.

The reader is referred to Alzer and Berg [2], Qi and Guo [20], and Qi, Guo and Chen [21] for proofs of this statement.

One goal of this paper is to study the Dirichlet $L$-functions from the point of view of logarithmically complete monotonicity. For $\text{Re } s > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. $$

Consider $s > 1$. Since $\log \zeta(s) > 0$ and

$$(-1)^k \frac{d^k}{ds^k} \log \zeta(s) = (-1)^k \frac{d^{k-1}}{ds^{k-1}} \left( \frac{\zeta'(s)}{\zeta(s)} \right) = \sum_{n=1}^{\infty} \frac{\Lambda(n) (\log n)^{k-1}}{n^s}, $$

where $\Lambda(n) \geq 0$ is the von Mangoldt function, $(-1)^k \frac{d^k}{ds^k} \log \zeta(s) > 0$ for all $s > 1$. This implies that $\zeta(s)$ is a logarithmically completely monotonic function for $s > 1$ (in fact, strictly logarithmically completely monotonic). But this approach fails in the case of $L(s, \chi)$ with $s > 1$ and $\chi$, a real primitive Dirichlet character modulo $q$, since

$$(-1)^k \frac{d^k}{ds^k} \log L(s, \chi) = (-1)^k \frac{d^{k-1}}{ds^{k-1}} \left( \frac{L'(s, \chi)}{L(s, \chi)} \right) = \sum_{n=1}^{\infty} \frac{\chi(n) \Lambda(n) (\log n)^{k-1}}{n^s}, $$

may change sign for different values of $s$ as $\chi(n)$ takes the values $-1, 0$ or 1. Hence, we need to consider a different approach for studying $L(s, \chi)$ in the context of logarithmically complete monotonicity. This naturally involves studying the zeros of derivatives of $\log L(s, \chi)$.

There have been several studies made on the number of zeros of $\zeta^{(k)}(s)$ and $L^{(k)}(s, \chi)$, one of which dates back to Spieser [23], who showed that the Riemann Hypothesis is equivalent to the fact that $\zeta'(s)$ has no zeros in $0 < \text{Re } s < 1/2$. Spira [24] conjectured that

$$N(T) = N_k(T) + \left\lceil \frac{T \log 2}{2\pi} \right\rceil \pm 1, $$

where $N_k(T)$ denotes the number of zeros of $\zeta^{(k)}(s)$ with positive imaginary parts up to height $T$, and $N(T) = N_0(T)$. Berndt [7] showed that for any $k \geq 1$, as $T \to \infty$,

$$N_k(T) = \frac{T \log T}{2\pi} - \left( \frac{1 + \log 4\pi}{2\pi} \right) T + O(\log T).$$

Levinson and Montgomery [18] proved a quantitative result implying that most of the zeros of $\zeta^{(k)}(s)$ are clustered about the line $\text{Re } s = 1/2$ and also showed that the Riemann Hypothesis implies that $\zeta^{(k)}(s)$ has at most finitely many non-real zeros in $\text{Re } s < 1/2$. Their results were further improved by Conrey and Ghosh [8]. Analogues of several of the above-mentioned results for Dirichlet $L$-functions were given by Yildirim [30]. Our results in this paper are related to the zeros of $\log L(s, \chi)$ and its derivatives.
Throughout the paper, we assume that $s$ is a real number and $\chi$ is a real primitive Dirichlet character modulo $q$. Let $F(s, \chi) := \log L(s, \chi)$, and for $s > 1$, define
\[
A_{\chi,k} := \{s : F^{(k)}(s, \chi) = 0\}. \tag{1.1}
\]
Then we obtain the following result:

**Theorem 1.1.** Let $\chi$ be a real primitive character modulo $q$ and $L(s, \chi) \neq 0$ for $0 < s < 1$. Then there exists a constant $c_\chi$ such that $[c_\chi, \infty) \cap (\cup_{k=1}^\infty A_{\chi,k})$ is dense in $[c_\chi, \infty)$.

Let us note that Theorem 1.1 shows in particular that $L(s, \chi)$ is not logarithmically completely monotonic on any subinterval of $[c_\chi, \infty)$. A stronger assertion is as follows:

For any subinterval of $[c_\chi, \infty)$, however small it may be, infinitely many derivatives $F^{(k)}(s, \chi)$ change sign in this subinterval.

Now consider any two points $s_1, s_2$ with $1 < s_1 < s_2$. In the case of the Riemann zeta function, if we compare the signs of the values of $\frac{d^k}{ds^k} \log \zeta(s)$ at $s_1$ and $s_2$ for all values of $k$, we see that they are always the same. Then a natural question arises - what can we say if we make the same comparison in the case of a Dirichlet $L$-function? We will see below that the answer is completely different (actually it is as different as it could be).

We first define a function $\psi_\chi$ for a real primitive Dirichlet character modulo $q$ as follows:

Let $B := \{g : \mathbb{N} \to \{-1,0,1\}\}$. Define an equivalence relation $\sim$ on $B$ by $g(n) = h(n)$ for all $n$ large enough. Let $\hat{B} = B / \sim$. By abuse of notation, we define $\psi_\chi : (1, \infty) \to \hat{B}$ to be a function whose image is a sequence given by $\{\text{sgn}(F^{(k)}(s, \chi))\}$, i.e.,
\[
\psi_\chi(s)(k) := \text{sgn}(F^{(k)}(s, \chi)). \tag{1.2}
\]

With this definition, we answer the above question in the form of the following theorem.

**Theorem 1.2.** Let $\chi$ be a real primitive character modulo $q$ and let $\psi_\chi$ be defined as above. Then there exists a constant $C_\chi$ with the following property:

(a) The Riemann hypothesis for $L(s, \chi)$ implies that $\psi_\chi$ is injective on $[C_\chi, \infty)$.

(b) Let $\psi_\chi$ be injective on $[C_\chi, \infty)$. Then there exists an effectively computable constant $D_\chi$ such that if all the nontrivial zeros $\rho$ of $L(s, \chi)$ up to the height $D_\chi$ lie on the critical line $\Re s = 1/2$, then the Riemann Hypothesis for $L(s, \chi)$ is true.

2. Proof of Theorem 1.1

First we will compute $F^{(k)}(s, \chi)$ in terms of the zeros of $L(s, \chi)$. The logarithmic derivative of $L(s, \chi)$ satisfies [11, page. 83]
\[
F'(s, \chi) = \frac{L'(s, \chi)}{L(s, \chi)} = -\frac{1}{2} \log \frac{q}{\pi} - \frac{1}{2} \Gamma'(s/2 + b/2) + B(\chi) + \sum_\rho \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right). \tag{2.1}
\]
where $B(\chi)$ is a constant depending on $\chi$,

$$b = \begin{cases} 
1 & \text{if } \chi(-1) = -1 \\
0 & \text{if } \chi(-1) = 1 
\end{cases},$$

(2.2)

and $\rho = \beta + i\gamma$ are the non trivial zeros of $L(s, \chi)$. Since $B(\overline{\chi}) = B(\chi)$ and $\chi$ is real, $B(\chi)$ is given by

$$B(\chi) = -\sum_{\rho \neq 0} \frac{1}{\rho} = -2 \sum_{\gamma > 0} \frac{\beta}{\beta^2 + \gamma^2} < \infty,$$

see [11, page. 83]. Note that $B(\chi)$ is negative. The Weierstrass infinite product for $\Gamma(s)$ is [11, p. 73]

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} (1 + s/n)^{-1} e^{s/n},$$

(2.3)

with $s = 0, -1, -2, \ldots$ being its simple poles. The functional equation for $\Gamma(s)$ is

$$\Gamma(s + 1) = s \Gamma(s)$$

(2.4)

$$\Gamma(s) \Gamma(s + 1/2) = 2^{(1-2s)} \pi^{1/2} \Gamma(2s),$$

(2.5)

where as the duplication formula for $\Gamma(s)$ is

$$\Gamma(s) \Gamma(s + 1/2) = 2^{(1-2s)} \pi^{1/2} \Gamma(2s),$$

(2.6)

see [11, p. 73]. The following can be easily derived from (2.4), (2.6) and the logarithmic derivative of (2.3):

$$\frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} = -\frac{\gamma}{2} - \frac{1}{s} - \sum_{n=1}^{\infty} \left( \frac{1}{s + 2n} - \frac{1}{2n} \right),$$

(2.7)

$$\frac{1}{2} \frac{\Gamma'(s/2 + 1/2)}{\Gamma(s/2 + 1/2)} = -\log(2) - \frac{\gamma}{2} - \sum_{n=0}^{\infty} \left( \frac{1}{s + 2n + 1} - \frac{1}{2n + 1} \right),$$

(2.8)

From (2.1), (2.2), (2.7) and (2.8), we have

$$F'(s, \chi) = -\frac{1}{2} \log \frac{q}{\pi} + b \log 2 + \frac{\gamma}{2} + B(\chi) + \frac{1-b}{s} + \sum_{\rho \neq 0} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right),$$

(2.9)

where $\rho$ runs through all the zeros of $L(s, \chi)$. The successive differentiation of (2.9) gives for $k \geq 2$,

$$F^{(k)}(s, \chi) = (-1)^{k-1}(k-1)! \left( \frac{1-b}{s^k} + \sum_{\rho \neq 0} \frac{1}{(s - \rho)^k} \right)$$

$$= (-1)^{k-1}(k-1)! \left( \sum_{L(\rho, \chi) = 0} \frac{1}{(s - \rho)^k} \right).$$

(2.10)
Let $s > 1/2$ and define
\[ l(s) := \min\{|s - \rho| : L(\rho, \chi) = 0\}. \tag{2.11} \]
If the minimum $l(s)$ is attained for a non-trivial zero $\rho$ of $L(s, \chi)$, then since the non-trivial zeros are symmetric with respect to the line $\sigma = 1/2$, we have $\Re \rho \geq 1/2$. Let $\tilde{\rho}_0$ be the non-trivial zero of $L(s, \chi)$ with minimum but positive imaginary part, i.e., $\Im \tilde{\rho}_0 = \min\{\Im \rho > 0 : L(\rho, \chi) = 0, \Re \rho \geq 1/2\}$. Write $\tilde{\rho}_0 = \tilde{\beta}_0 + i\tilde{\gamma}_0$. Then for all $s > \tilde{\gamma}_0^2 + 1/4$, we have $s^2 > (s - 1/2)^2 + \tilde{\gamma}_0^2 \geq |s - \tilde{\rho}_0|^2 \geq (l(s))^2$. Define
\[ c_{\chi} := \inf\{c > 1 : s > c \Rightarrow |s| > l(s)\}. \tag{2.12} \]
The constant $c_{\chi}$ is defined in this way since we want $l(s)$ to be attained at a non-trivial zero of $L(s, \chi)$, as this will allow us to separate the two terms of the series in (2.10) corresponding to this zero and its conjugate, which together will give a dominating term essential in the proof. Note that if $\tilde{\gamma}_0 \leq \sqrt{3}/2$, $c_{\chi} = 1$, otherwise $1 \leq c_{\chi} \leq \tilde{\gamma}_0^2 + 1/4$.

Next we show that for any $s \geq c_{\chi}$, there is an $s' \in (s - \epsilon, s + \epsilon)$, $\epsilon > 0$, so that $l(s')$ is attained at a unique non-trivial zero $\rho'$ of $L(s, \chi)$ with $\Im \rho' > 0$.

For any real number $s_0 > c_{\chi}$, consider the interval $(s_0 - \epsilon, s_0 + \epsilon) \subset [c_{\chi}, \infty)$ for some $\epsilon > 0$. Let
\[ A := \{\rho' : \Im \rho' \geq 0 \text{ and } |s_0 - \rho'| = l(s_0), L(\rho', \chi) = 0\}, \tag{2.13} \]
that is, $A$ is comprised of all non-trivial zeros on the circle with center $s_0$ and radius $l(s_0)$. Clearly $A$ is a finite set since $|A| \leq N(l(s_0), \chi)$, where $N(T, \chi)$ denotes the number of zeros of $L(s, \chi)$ up to height $T$. As shown in Figure 1, let $\rho_0 \in A$ with $\Re \rho_0 = \max\{\Re \rho : \rho \in A\}$. Then for any $s \in (s_0, s_0 + \epsilon)$, $|s - \rho_0| < |s - \rho|$, for all $\rho \in A$, $\rho \neq \rho_0$.

![Figure 1. Construction for identifying the unique $\rho_0$ at which $l(s)$ is attained for $s \in (s_0 - \epsilon, s_0 + \epsilon)$.](image-url)
\[ \text{Re } \rho_1 = \min \{ \text{Re } \rho : |s_1 - \rho_0| = |s_1 - \rho|, \rho \neq \rho_0, L(\rho, \chi) = 0 \}. \]

Note that \( \text{Im } \rho_1 > \text{Im } \rho_0 \), otherwise it will contradict the fact that the minimum \( l(s_0) \) is attained at \( \rho_0 \).

For any \( s \in (s_0, s_1) \), \( |s - \rho_0| < |s - \rho_1| \). Now fix one such \( s \), say \( s_2 \in (s_0, s_1) \), and find a \( \rho_2 \) so that \( \text{Re } \rho_2 = \min \{ \text{Re } \rho : |s_2 - \rho_0| = |s_2 - \rho|, \rho \neq \rho_0, L(\rho, \chi) = 0 \} \). Since there are only finite many zeros in the rectangle \([0, 1] \times [\text{Im } \rho_0, \text{Im } \rho_1]\), repeating the argument allows us to find an \( s' \in \mathbb{R} \) and \( s_0 < s' < s_1 \), so that \( \rho_0 \) is the only non-trivial zero of \( L(s, \chi) \) with \( \text{Im } \rho_0 \geq 0 \) and \( |s' - \rho_0| = \min \{|s' - \rho|, \text{Im } \geq 0 \text{ and } L(\rho, \chi) = 0\} \), i.e., the circle with center \( s' \) and radius \( |s - \rho_0| \) does not contain any zero other than \( \rho_0 \) itself. Note that for any \( s \in (s_0, s') \), \( \rho_0 \) is the only zero at which \( l(s) \) is attained.

Next, let \( B = \{ \rho' : \rho' \neq \rho_0, |s_0 - \rho'| < |s_0 - \rho| \} \), where \( \rho, \rho' \) are zeros of \( L(s, \chi) \). Note that \( B \) is also a finite set. Arguing in a similar way as above, we can find a \( \tilde{\rho} \in B \) and \( s'' \in (s_0, s_0 + \epsilon) \) so that for all \( s \in (s_0, s'') \), \( |s - \tilde{\rho}| \leq |s - \rho| \) for \( \rho \neq \rho_0 \).

Therefore we can find a closed interval \([c, d] \subset (s_0 - \epsilon, s_0 + \epsilon)\) so that for all \( s \in [c, d] \), we have

\[ l(s) = |s - \rho_0| = |s - \tilde{\rho}| < |s - \rho|, \rho \neq \rho_0, \rho_0, \tilde{\rho}. \]

Now let \( s - \rho_0 = r_s e^{i\theta_s} \) for all \( c \leq s \leq d \). Then from (2.10) and the fact that the zeros of \( L(s, \chi) \) are symmetric with respect to the real axis, we have

\[
F^{(k)}(s, \chi) = (-1)^{k-1}(k-1)! \left( \frac{1}{(s-\rho_0)^k} + \frac{1}{(s-\rho_0)^k} + \sum_{\rho \neq \rho_0} \frac{1}{(s-\rho)^k} \right) \\
= (-1)^{k-1}(k-1)! \left( \frac{2}{r_s^k} \cos(k\theta_s) + \sum_{\rho \neq \rho_0, \rho_0} \frac{1}{(s-\rho)^k} \right) \\
= \frac{(-1)^{k-1}(k-1)!}{r_s^k} \left( 2 \cos(k\theta_s) + f(s) \right),
\]

where \( f(s) := \frac{1}{r_s^k} \sum_{\rho \neq \rho_0, \rho_0} \frac{1}{(s-\rho)^k} \) and \( k \geq 2 \). Since the series \( \sum_{\rho \neq \rho_0, \rho_0} \frac{1}{(s-\rho)^k} \) converges absolutely for \( k \geq 1 \), \( f(s) \) is a differentiable function for \( s > 1 \). Now,

\[ |f(s)| \leq 2 \sum_{\rho \neq \rho_0, \rho_0} \frac{r_s^k}{|s-\rho|^k} = 2 \sum_{\rho \neq \rho_0, \rho_0, \rho_0} \frac{|s-\rho_0|^k}{|s-\rho|^k} \\
= 2|s - \rho_0|^2 \sum_{\rho \neq \rho_0, \rho_0, \rho_0} \frac{1}{|s-\rho|^2} |s-\rho_0|^{k-2} \\
\leq 2|s - \rho_0|^2 \sum_{\rho \neq \rho_0, \rho_0, \rho_0} \frac{1}{|s-\rho|^2} |s-\rho_0|^{k-2} \\
\leq 2|s - \rho_0|^2 \sum_{\rho \neq \rho_0, \rho_0, \rho_0} \frac{1}{|s-\rho|^2} \sup_{c \leq s \leq d} \left\{ \frac{|s-\rho_0|^{k-2}}{|s-\rho_0|^{k-2}} \right\}, \quad (2.17)
\]
where in the penultimate step we use (2.15). Let $h(s) := \frac{|s - \rho_0|}{|s - \rho|}$. Then $h(s)$ is a continuous function on $[c, d]$ and hence attains its supremum on $[c, d]$. Thus there exists an $x \in [c, d]$ such that

$$
\eta := \sup_{c \leq x \leq d} \left\{ \frac{|s - \rho_0|}{|s - \rho|} \right\} = \frac{|x - \rho_0|}{|x - \rho|}.
$$

(2.18)

Therefore by (2.14), $\eta < 1$. Combining (2.17) and (2.18), we have

$$
|f(s)| \leq 2\eta^{k-2}|s - \rho_0|^2 \sum_{\rho \neq \rho_0, \rho_0 \in \mathbb{L}, \Im \rho \geq 0} \frac{1}{|s - \rho|^2} \leq 2\eta^{k-2}|d - \rho_0|^2 \sum_{\rho \neq \rho_0, \rho_0 \in \mathbb{L}, \Im \rho \geq 0} \frac{1}{|c - \rho|^2} \leq C_{c, d, \chi} \eta^{k-2}.
$$

(2.19)

Note that the constant term depends only on $c, d$ and $\chi$. Hence for sufficiently large $k$, we have $|f(s)| < 1$. Let $c - \rho_0 = r_c e^{i\theta_c}$ and $d - \rho_0 = r_d e^{i\theta_d}$. Then $\theta_c > \theta_d$. For $k$ large enough, we can write $2\pi < k(\theta_c - \theta_d)$. Since for $s \in [c, d]$, we have $\theta_d \leq \theta_s \leq \theta_c$, for a sufficiently large $k$, $\cos(k\theta_s)$ attends all the values of the interval $[-1, 1]$. So from (2.17) and (2.19) we conclude that for each large enough $k$ there will be an $s$ in $[c, d] \subset (s_0 - \epsilon, s_0 + \epsilon)$ so that $F(k)(s, \chi) = 0$. This shows that $\bigcup_{k=1}^{\infty} A_{\chi, k}$ has a non-empty intersection with $(s_0 - \epsilon, s_0 + \epsilon)$ for any $s_0 > c_{\chi}$. This completes the proof of the theorem.

**Remark:** Let $\chi$ be a real nonprincipal Dirichlet character. If $L(s, \chi)$ has a Siegel zero, call it $\beta$, and if every zero of $L(s, \chi)$ has real part $\leq \beta$, then for any $s > 1$, (2.10) implies

$$
F(k)(s, \chi) = \frac{(-1)^{k-1}(k-1)!}{(s - \beta)^k} \left( 1 + \sum_{\rho \neq \beta, \Im \rho = 0} \left( \frac{s - \beta}{s - \rho} \right)^k \right).
$$

(2.20)

Arguing as in the proof of Theorem 1.1, we see that there exists an integer $M$ such that for all $k \geq M$, the series in (2.20) is less than 1. This means that for those $k$, $F(k)(s, \chi)$ maintains the same sign for all $s > 1$. This is why we include the condition that $L(s, \chi) \neq 0$ for $0 < s < 1$ in the hypotheses of Theorem 1.1.

3. **Proof of Theorem 1.2**

Assume that the Riemann hypothesis holds for $L(s, \chi)$. Let $\gamma_0 := \Im \rho_0 = \min\{\Im \rho \geq 0 : L(\rho, \chi) = 0\}$, where $\rho_0, \rho$ are non-trivial zeros of $L(s, \chi)$. Then $\rho_0 = 1/2 + i\gamma_0$. We show that the function $\psi_{\chi}$ is injective on $[C_{\chi}, \infty)$, where the constant $C_{\chi}$ will be determined later.
Let $s > c_\chi$, where $c_\chi$ is defined in (2.12). Then $l(s) < |s|$ and $l(s) = |s - \rho_0| < |s - \rho|$ for $\rho \neq \rho_0, \tilde{\rho}_0$. Let $s - \rho_0 = r_s e^{i\theta_s}$. From (2.16), we have for $k \geq 2$,

$$|f(s)| \leq \sum_{\rho \neq \rho_0, \tilde{\rho}_0} \frac{n_s^k}{|s - \rho|^k} = |s - \rho_0|^2 \sum_{\rho \neq \rho_0, \tilde{\rho}_0} \frac{1}{|s - \rho|^2} \left| \frac{|s - \rho_0|^{k-2}}{|s - \rho|^{k-2}} \right|

\leq |s - \rho_0|^2 \sum_{\rho \neq \rho_0, \tilde{\rho}_0} \frac{1}{|s - \rho|^2} \sup_{\rho} \left\{ \left| \frac{|s - \rho_0|^{k-2}}{|s - \rho|^{k-2}} \right| \right\}

= |s - \rho_0|^2 \eta_s^{k-2} \sum_{\rho \neq \rho_0, \tilde{\rho}_0} \frac{1}{|s - \rho|^2}

= O_{s, \chi}(\eta_s^{k-2}). \quad (3.1)$$

Here in the penultimate step,

$$\eta_s = \sup_{\rho} \left\{ \frac{|s - \rho_0|}{|s - \rho|} \right\} \leq \frac{|s - \rho_0|}{|s - \tilde{\rho}|} < 1,$$

and $\Im \rho_0 < \Im \tilde{\rho} \leq \Im \rho$, resulting from (2.14) and (2.15). Combining (2.16) and (3.1), we obtain

$$F^{(k)}(s, \chi) = \frac{2(-1)^{k-1}(k-1)!}{n_s^k} \left( \cos(k\theta_s) + f(s) \right), \quad (3.2)$$

where $f(s) = O_{s, \chi}(\eta_s^{k-2})$.

Next, we show that there are infinitely many $k$ for which $\cos(k\theta_s)$, which we view as the main term, dominate the error term. Since $\eta_s < 1$, for a fixed $s > 1$, we can bound the error term in $(-\epsilon, \epsilon)$ for all sufficiently large $k$ and for all $0 < \epsilon < 1$. Write $\cos(k\theta_s) = \cos \left( \frac{\pi k\theta_s}{2\pi} \right) = \cos \left( 2\pi \frac{k\theta_s}{2\pi} \right)$ and consider the cases when $\frac{\theta_s}{\pi}$ is rational and $\frac{\theta_s}{2\pi}$ is irrational.

If $\frac{\theta_s}{\pi}$ is a rational number, there are infinitely many $k \in \mathbb{N}$ so that $\frac{k\theta_s}{2\pi}$ is an even integer and hence $\cos(k\theta_s) = 1$.

If $\frac{\theta_s}{\pi}$ is a rational number with odd numerator, then there are infinitely many $k \in \mathbb{N}$, namely the odd multiples of the denominator, so that $\frac{k\theta_s}{2\pi}$ is an odd integer and hence $\cos(k\theta_s) = -1$.

Let $\frac{\theta_s}{2\pi} = \frac{2m}{n}$ be a rational number with even numerator and odd denominator. Since $(2m, n) = 1$, there exists an integer $l \in [1, n]$ such that $2ml \equiv 1 \pmod{n}$. For all $k \equiv l \pmod{n}$, $2mk \equiv 1 \pmod{n}$. Therefore for all $k \equiv l \pmod{n}$, since $2mk$ is even, we have $2mk = (2p + 1)n + 1$. Hence there are infinitely many integers $k$ for which $\cos(k\theta_s) = \cos \left( \pi \left( 2p + 1 + \frac{1}{n} \right) \right) = -\cos \left( \frac{\pi}{n} \right)$.

If $\frac{\theta_s}{2\pi}$ is irrational, then we know [28] that the sequence $\left\{ \left\{ \frac{k\theta_s}{2\pi} \right\} \right\}$ is dense in $[0, 1]$, where $\{x\}$ denotes the fractional part of $x$. Hence there are infinitely many $k \in \mathbb{N}$ with $\left\{ \frac{k\theta_s}{2\pi} \right\}$ close to 1 and hence $\cos(k\theta_s) > 1 - \epsilon$ for any given $\epsilon > 0$. Likewise, there are infinitely many $k \in \mathbb{N}$ with $\left\{ \frac{k\theta_s}{2\pi} \right\}$ close to $\frac{1}{2}$ and hence $\cos(k\theta_s) < -1 + \epsilon$.

Fix $s_1$ and $s_2$ such that $c_\chi < s_1 < s_2$. Then $l(s_1) = |s_1 - \rho_0|$ and $l(s_2) = |s_2 - \rho_0|$. Let $\theta_1$ and $\theta_2$ be such that $s_1 - \rho_0 = r_1 e^{i\theta_1}$ and $s_2 - \rho_0 = r_2 e^{i\theta_2}$. Note that $0 < \theta_2 < \theta_1 < \pi/2$. 
From (3.2), we have
\[ F(k)(s_1, \chi) = \frac{2(-1)^{k-1}(k-1)!}{\tau_1^k} (\cos(k\theta_1) + f(s_1)), \]  
(3.3)
\[ F(k)(s_2, \chi) = \frac{2(-1)^{k-1}(k-1)!}{\tau_2^k} (\cos(k\theta_2) + f(s_2)), \]  
(3.4)
where \( f(s_1) = O_{s_1, \chi}(\eta_{s_1}^{k-2}) \) and \( f(s_2) = O_{s_2, \chi}(\eta_{s_2}^{k-2}) \). Write \( \theta_1 = \theta_2 + (\theta_1 - \theta_2) \).

We show that there exist infinitely many integers \( k \) such that the terminal rays of \( k\theta_1 \) and \( k\theta_2 \) stay away from the \( y \)-axis, that \( \text{sgn} (\cos k\theta_1) = -\text{sgn} (\cos k\theta_2) \neq 0 \), and that \( \cos(k\theta_1) \) and \( \cos(k\theta_2) \) dominate \( f(s_1) \) and \( f(s_2) \) in (3.3) and (3.4) respectively. We first determine the signs.

Case 1: If \( \frac{\theta_1 - \theta_2}{\pi} \) is rational with odd numerator then as we saw before, there are infinitely many positive integers \( k \) so that \( k\frac{\theta_1 - \theta_2}{\pi} \) is an odd integer and hence for those \( k \in \mathbb{N} \), \( \cos(k\theta_1) = \cos(k\theta_2 + \pi) = -\cos(k\theta_2) \).

Case 2: If \( \frac{\theta_1 - \theta_2}{\pi} \) is rational with even numerator and odd denominator \( n \), there are infinitely many positive integers \( k \) so that \( k\frac{\theta_1 - \theta_2}{\pi} = 2p + 1 + \frac{1}{n} \) for some \( p \in \mathbb{N} \) and so \( \cos(k\theta_1) = \cos(k\theta_2 + \pi + \frac{1}{n}) = -\cos(k\theta_2 + \pi \frac{1}{n}) \).

Case 3: If \( \frac{\theta_1 - \theta_2}{2\pi} \) is irrational, there are infinitely many positive integers \( k \) such that \( \left\{ k\frac{\theta_1 - \theta_2}{2\pi} \right\} \in [1/2, 1/2 + \epsilon/2\pi) \), for any given \( \epsilon > 0 \). So for any \( \delta \) such that \( 0 < \delta < \epsilon \), we have \( \cos(k\theta_1) = \cos(k\theta_2 + \pi + \delta) = -\cos(k\theta_2 + \delta) \). We can choose \( \epsilon \) as small as we want and hence \( 0 < \delta < \epsilon < \pi/n \).

We first show that in Case 2, we have the terminal rays of the angles sufficiently away from the \( y \)-axis, with \( \cos k\theta_1 \) and \( \cos k\theta_2 \) dominating their corresponding terms \( f(s_1) \) and \( f(s_2) \). To that end, choose a constant \( b_\chi > 1/2 \) such that \( \tan \left( \frac{\pi}{100} \right) = \frac{\eta_0}{b_\chi - \frac{1}{2}} \), say. If \( s - \rho_0 = r_s e^{i\theta_2} \) and \( s > b_\chi \), then \( 0 < \theta_s < \pi/100 \). So if we take \( b_\chi < s_1 < s_2 \), then \( 0 < \theta_2 < \theta_1 < \pi/100 \). Since \( \eta_1, \eta_2 < 1 \) there exists an integer \( K \) such that \( |f(s_1)|, |f(s_2)| < \theta_2/4 \) for all \( k > K \). As we saw before, for infinitely many integers \( k > K + 2 \), we have \( k\theta_1 = k\theta_2 + \pi + \pi/n \), where \( n \) depends on \( \theta_1 \) and \( \theta_2 \). We first note that all angles below are considered mod \( 2\pi \). If \( k\theta_2 \in (\pi/2 + \theta_2, \pi) \) then \( k\theta_1 \in (\pi/2 + \theta_2, 2\pi - \theta_2) \). Thus \( \cos(k\theta_1) \cos(k\theta_2) < 0 \). Also \( |\cos(k\theta_2)| > |\sin(\theta_2)| \geq \theta_2/2 > |f(s_2)| \) and \( |\cos(k\theta_1)| = |\cos(k\theta_2 + \pi/n)| > |\sin(\theta_2)| \geq \theta_2/2 > |f(s_1)| \).

Similarly we see that \( |\cos(k\theta_1)| > |f(s_1)| \) and \( |\cos(k\theta_2)| > |f(s_2)| \) when \( k\theta_2 \in (\pi/2 + \theta_2, 0) \). If \( k\theta_2 \in (0, \pi/2 - \theta_2) \) and \( k\theta_1 \in (-\pi, -\pi/2 - \theta_2) \) in this case also \( |\cos(k\theta_2)| > |\sin(\theta_2)| \geq \theta_2/2 > |f(s_2)| \) and \( |\cos(k\theta_1)| = |\cos(k\theta_2 + \pi/n)| > |\sin(\theta_2)| \geq \theta_2/2 > |f(s_1)| \). Now let \( k\theta_2 \in (0, \pi/2 + \theta_2) \) and \( k\theta_1 \in (-\pi/2 - \theta_2, 0) \). Then since \( \pi/n < \theta_1 < \pi/100 \), it is easy to check that \( (k-2)\theta_2 \in (0, \pi/2 - \theta_2) \) and \( (k-2)\theta_1 = k\theta_2 + \pi + \pi/n - 2\theta_1 \in (-\pi, -\pi/2 - \theta_2) \). Hence \( |\cos(k\theta_1)| > |f(s_1)| \) and \( |\cos(k\theta_2)| > |f(s_2)| \). Similarly we have the same conclusion if \( k\theta_2 \in (-\pi, -\pi/2 + \theta_2) \) and \( k\theta_1 \in (\pi/2 - \theta_2, \pi) \).

Let \( C_\chi = \max\{c_\chi, b_\chi\} \).
Then for any given real numbers \( s_1 \) and \( s_2 \) such that \( C_\chi < s_1 < s_2 \), we have shown that there exist infinitely many integers \( k \) such that \( \cos(k\theta_1) \) and \( \cos(k\theta_2) \) have opposite signs and \( |\cos(k\theta_1)| > |f(s_1)| \) and \( \cos(k\theta_2) > f(s_2) \). This implies that \( F^{(k)}(s_1, \chi) \) and \( F^{(k)}(s_2, \chi) \) have opposite signs and that in turn proves that the function \( \psi_\chi \) is injective in \([C_\chi, \infty)\).

We now prove part (b) of Theorem 1.2.

\[
\begin{align*}
\text{Figure 2. Constructing the angle } \phi = 2\pi(a + b\sqrt{2}).
\end{align*}
\]

Let \( \rho_0 \) be the lowest zero of \( L(s, \chi) \) above the real axis (so \( \rho_0 \) is not a real number). Let \( L_1 \) be the line passing through \( \rho_0 \) and perpendicular to the line which passes through \( \rho_0 \) and \( C_\chi \), where \( C_\chi \) is defined in (3.5). Let \( (1, D_\chi) \) be the point of intersection of the lines \( \sigma = 1 \) and \( L_1 \). We first show that if there is only one zero \( \rho_1 \) with \( \text{Im} \rho_1 \geq D_\chi \) off the critical line \( \sigma = 1/2 \), then this contradicts the injectivity of \( \psi_\chi \) on \([C_\chi, \infty)\).

Without loss of generality, let \( \text{Re} \rho_1 > 1/2 \). As shown in Figure 2, let \( L_2 \) be the line passing through \( \rho_0 \) and \( \rho_1 \). Let \( s_0 \) and \( s_1 \) be the points of intersection of the real axis with the lines perpendicular to \( L_2 \) and passing through \( \rho_0 \) and \( \rho_1 \) respectively. Clearly \( s_1 > s_0 > C_\chi \). Note that by our construction, \( l(s_0) = |s_0 - \rho_0| \) and \( l(s_1) = |s_1 - \rho_1| \), where \( l(s) \) is defined in (2.11), and there exists a \( \theta \) such that \( (s_0 - \rho_0) = r_s e^{i\theta} \) and \( (s_1 - \rho_1) = r_s e^{i\theta} \). From the proof of the Theorem 1.1, we know that there exists an \( \epsilon > 0 \) so that \( l(s) = |s - \rho_0| \) for all \( s \in (s_0 - \epsilon, s_0 + \epsilon) \) and \( l(s) = |s - \rho_1| \) for all \( s \in (s_1 - \epsilon, s_1 + \epsilon) \). Without loss of generality, we can assume that \( s_0 + \epsilon < s_1 - \epsilon \).

Therefore, there exists a \( \delta > 0 \) such that \( \theta_s \in (\theta - \delta, \theta + \delta) \), where \( s - \rho_0 = r_s e^{i\theta_s} \), and \( l(s) = |s - \rho_0| \) for all \( s \in (s_0 - \epsilon, s_0 + \epsilon) \), and \( \theta_s \in (\theta - \delta, \theta + \delta) \), where \( s - \rho_1 = r_s e^{i\theta_s} \), and \( l(s) = |s - \rho_1| \) for all \( s \in (s_1 - \epsilon, s_1 + \epsilon) \).

Since the sequence \( \{n\sqrt{2}\} \) is dense in \([0, 1)\), and \( \{n\sqrt{2}\} = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor \), there exists an integer \( a \) and an integer \( b \neq 0 \) such that \( a + b\sqrt{2} \in (\theta - \delta, \theta + \delta) \). Let \( \phi = 2\pi(a + b\sqrt{2}) \), \( s' \in (s_0 - \epsilon, s_0 + \epsilon) \) and \( s'' \in (s_1 - \epsilon, s_1 + \epsilon) \) be such that \( s' - \rho_0 = r_s e^{i\phi} \).
and \( s'' - \rho_1 = r_{s''}e^{i\phi} \). Therefore,

\[
F^{(k)}(s', \chi) = \frac{2(-1)^{k-1}(k-1)!}{r_{s'}^k} (\cos(k\phi) + f(s'))
\]

(3.6)

\[
F^{(k)}(s'', \chi) = \frac{2(-1)^{k-1}(k-1)!}{r_{s''}^k} (\cos(k\phi) + f(s'')) ,
\]

(3.7)

where \(|f(s')| = O(\eta_{s'}^{k-2})\) and \(|f(s'')| = O(\eta_{s''}^{k-2})\). Let \( \eta = \min\{\eta_{s'}, \eta_{s''}\} \). Then \(|f(s')|, |f(s'')| \leq C_{s', s''}\eta^{k-2}\) for some constant \( C_{s', s''} \).

We next show that there exist positive constants \( C_{a, b} \) and \( K_{a, b} \) so that

\[
|4k(a + b\sqrt{2}) + r| > \frac{C_{a, b}}{k},
\]

(3.8)

for any integers \( r \) and \( k \), with \( k > K_{a, b} \). Let \(|4k(a + b\sqrt{2}) + r| \leq 1 \). Then,

\[
|4k(a - b\sqrt{2}) + r| \leq |4k(a + b\sqrt{2}) + r| + 8k|b|\sqrt{2} \leq 1 + 8k|b|\sqrt{2} < \frac{k}{C_{a, b}}.
\]

(3.9)

Therefore for \( k \geq 2 \),

\[
|4k(a + b\sqrt{2}) + r|\frac{k}{C_{a, b}} > |4k(a - b\sqrt{2}) + r|\frac{k}{C_{a, b}} = |(4ka + r) - 2(4kb)^2| \geq 1,
\]

(3.10)

since \( b \neq 0 \). If \(|4k(a + b\sqrt{2}) + r| \geq 1 \), then of course, there exists a \( K_{a, b} \), such that for \( k > K_{a, b} \), we have \(|4k(a + b\sqrt{2}) + r| > \frac{C_{a, b}}{k} \). Hence in conclusion, for a large positive integer \( N \) and for all \( k > N \), if we choose \( m \) so that \(|4k(a + b\sqrt{2}) \pm 1 \pm 4m| < 1 \), we have

\[
|\cos k\phi| = \left|\sin \frac{\pi}{2}(4k(a + b\sqrt{2}) \pm 1 \pm 4m)\right| \geq \sin \left(\frac{\pi C_{a, b}}{4k}\right)
\]

\[
\geq \frac{\pi C_{a, b}}{4k}
\]

\[
> C_{s', s''}\eta^{k-2}.
\]

(3.11)

Therefore for the above mentioned \( s' \) and \( s'' \) such that \( s' \neq s'' \), and for all \( k > N \), \( F^{(k)}(s', \chi) \) and \( F^{(k)}(s'', \chi) \) have the same sign. This contradicts the injectivity of \( \psi_\chi \) on \([C_\chi, \infty)\). Now if there is more than one zero \( \rho \) with \( \text{Im}\rho \geq D_\chi \) off the critical line, then we can choose the zero \( \rho_1 \) with the following properties:

\( i \) The angle between the positive \( x \)-axis and the line \( L \) passing through the zeros \( \rho_0 \) and \( \rho_1 \) is smaller than the angle between the positive \( x \)-axis and the line passing through the zeros \( \rho_0 \) and \( \rho \neq \rho_1 \) and,

\( ii \) \( \text{Im}\rho_1 = \min\{\text{Im}\rho \geq D_\chi : \rho \text{ lies on the line } L\} \).

Then we can proceed similarly as above and again get a contradiction. Hence, all the zeros above the line \( t = D_\chi \) lie on the critical line \( \sigma = 1/2 \). This completes the proof.

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