RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

RECURRENCES

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GUESS AND PROVE

$T(n) = cn$ (WE WILL FIND THE VALUE OF c)

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) = cn$ (WE WILL FIND THE VALUE OF c)

BASE CASE: $T(1) = 1 = c$ (ASSUME $c \geq 1$)

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BASE CASE: $T(1) = 1 = c$ (ASSUME $c \geq 1$)

INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

TPT: $T(n) = cn$

RECURRENCES

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INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

<u>T.P.T</u> : $T(n) = cn$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$T(1) = 1$

GUESS AND PROVE

$T(n) \quad cn$ (WE WILL FIND THE VALUE OF c)

BASE CASE : $T(1) = 1 = c$ (ASSUME $c \geq 1$)

INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

TPT : $T(n) = cn$

$$T(n) = 4\underbrace{T\left(\frac{n}{2}\right)}_{\text{USE INDUCTION HYPOTHESIS}} + n$$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) = cn$ (WE WILL FIND THE VALUE OF c)

BASE CASE : $T(1) = 1 = c$ (ASSUME $c \geq 1$)

INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

<u>TPT</u>: $T(n) = cn$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$= 4 \cdot \frac{cn}{2} + n$$

$$= 2cn + n$$

$$= n(2c+1)$$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

WRONG GUESS AND PROVE

GUESS $T(n) = cn$ (WE WILL FIND THE VALUE OF c)

BASE CASE : $T(1) = 1 = c$ (ASSUME $c \geq 1$)

INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

TPT : $T(n) = cn$

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + n \\ &= 4 \cdot \frac{cn}{2} + n \\ &= 2cn + n \\ &= n(2c+1) \\ &> cn \end{aligned}$$

CANNOT SAY THAT THIS IS $O(n)$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) \le cn^3$ (WE WILL FIND THE VALUE OF c)

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) \leq cn^3$ (WE WILL FIND THE VALUE OF c)

BASE CASE: $T(1) = 1 \leq c$ $(c \geq 1)$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) \leq cn^3$ (WE WILL FIND THE VALUE OF c)

BASE CASE: $T(1) = 1 \leq c$ $(c \geq 1)$

INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

T.P.T $T(n) \leq cn^3$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

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INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

T.P.T $T(n) \leq cn^3$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$= 4c\frac{n^3}{8} + n$$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) \leq cn^3$ (WE WILL FIND THE VALUE OF c)

BASE CASE: $T(1) = 1 \leq c$ $(c \geq 1)$

INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

T.P.T $T(n) \leq cn^3$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$= 4c\frac{n^3}{8} + n$$

$$= \frac{cn^3}{2} + n$$

$$= cn^3\left(\frac{1}{2} + \frac{1}{cn^2}\right)$$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) \le cn^3$ (WE WILL FIND THE VALUE OF c)

BASE CASE: $T(1) = 1 \le c$ $(c \ge 1)$

INDUCTION HYPOTHESIS: THE STATEMENT IS TRUE FOR $T(1), T(2), \dots, T(n-1)$

T.P.T $T(n) \le cn^3$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$= 4c\frac{n^3}{8} + n$$

$$= \frac{cn^3}{2} + n$$

$$= cn^3 \underbrace{\left(\frac{1}{2} + \frac{1}{cn^2}\right)}$$

NEED TO SHOW THIS TERM ≤ 1 FOR $n \ge 2$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) \le cn^3$ (WE WILL FIND THE VALUE OF c)

BASE CASE: $T(1) = 1 \le c$ $(c \ge 1)$

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T.P.T $T(n) \le cn^3$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

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$$= \frac{cn^3}{2} + n$$

$$= cn^3 \underbrace{\left(\frac{1}{2} + \frac{1}{cn^2}\right)}$$

NEED TO SHOW THIS TERM ≤ 1 FOR $n \ge 2$

$$\boxed{c = 1}$$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

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GUESS AND PROVE

$T(n) \le cn^3$ (WE WILL FIND THE VALUE OF c)

BASE CASE: $T(1) = 1 \le c$ $(c \ge 1)$

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$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

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NEED TO SHOW THIS TERM ≤ 1 FOR $n \ge 2$

$\boxed{c = 1}$

$$\le cn^3$$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

GUESS AND PROVE

$T(n) \le cn^2$ (WE WILL FIND THE VALUE OF c)

RECURRENCES

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T.P.T: $T(n) \le cn^2$

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$T(n) \leq cn^2$ (WE WILL FIND THE VALUE OF c)

BASE CASE: $T(1) = 1 \leq c$ (ASSUME $c \geq 1$)

INDUCTION HYPOTHESIS: ASSUME THAT THE STATEMENT IS TRUE FOR $T(1), T(2), \ldots, T(n-1)$

T.P.T: $T(n) \leq cn^2$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$= 4\frac{cn^2}{4} + n$$

$$= cn^2 + n$$

$$\not\leq cn^2$$

RECURRENCES

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(1) = 1$$

$$T(n) = 2n^2 - n$$
$$= O(n^2)$$

MASTER'S THEOREM:

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

$$T(1) = c$$

WHERE $a \geq 1$, $b \geq 2$, $c \geq 0$, $d \geq 1$

THEN $T(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$

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EXAMPLES: (1) $T(n) = 2T\left(\frac{n}{2}\right) + n$

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EXAMPLES : (1) $T(n) = 2T\left(\frac{n}{2}\right) + n$

$a = 2 \quad b = 2 \quad d = 1$

$a = b^d$

$T(n) = O(n \log n)$

MASTER'S THEOREM :

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

$$T(1) = c$$

WHERE $a \geq 1,\ b \geq 2,\ c \geq 0,\ d \geq 1$

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EXAMPLES : (1) $T(n) = 2T\left(\frac{n}{2}\right) + n$

$a = 2 \quad b = 2 \quad d = 1$

$a = b^d$

$$T(n) = O(n \log n)$$

(2) $T(n) = 3T\left(\frac{n}{2}\right) + n$

MASTER'S THEOREM:

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$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

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WHERE $a \geq 1$, $b \geq 2$, $c \geq 0$, $d \geq 1$

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EXAMPLES: (1) $T(n) = 2T\left(\frac{n}{2}\right) + n$

$a = 2$ $b = 2$ $d = 1$

$a = b^d$

$T(n) = O(n \log n)$

(2) $T(n) = 3T\left(\frac{n}{2}\right) + n$

$a = 3$ $b = 2$ $d = 1$

$a > b^d$

MASTER'S THEOREM:

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

$$T(1) = c$$

WHERE $a \geq 1,\ b \geq 2,\ c \geq 0,\ d \geq 1$

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EXAMPLES: (1) $T(n) = 2T\left(\frac{n}{2}\right) + n$

$a = 2 \quad b = 2 \quad d = 1$

$a = b^d$

$T(n) = O(n \log n)$

(2) $T(n) = 3T\left(\frac{n}{2}\right) + n$

$a = 3 \quad b = 2 \quad d = 1$

$a > b^d$

$T(n) = O(n^{\log_2 3})$

MASTER'S THEOREM:

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

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WHERE $a \geq 1,\ b \geq 2,\ c \geq 0,\ d \geq 1$

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$$T(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

EXAMPLES: (1) $T(n) = 2T\left(\frac{n}{2}\right) + n$

$a = 2 \quad b = 2 \quad d = 1$

$$a = b^d$$

$$T(n) = O(n \log n)$$

(2) $T(n) = 3T\left(\frac{n}{2}\right) + n$

$a = 3 \quad b = 2 \quad d = 1$

$$a > b^d$$

$$T(n) = O(n^{\log_2 3})$$

(3) $T(n) = T(n/2) + n$

MASTER'S THEOREM:

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$
$$T(1) = c$$

WHERE $a \geq 1,\ b \geq 2,\ c \geq 0,\ d \geq 1$

THEN
$$T(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

EXAMPLES: (1) $T(n) = 2T\left(\frac{n}{2}\right) + n$

$a = 2 \qquad b = 2 \qquad d = 1$

$$a = b^d$$

$$T(n) = O(n \log n)$$

(2) $T(n) = 3T\left(\frac{n}{2}\right) + n$

$a = 3 \qquad b = 2 \qquad d = 1$

$$a > b^d$$

$$T(n) = O\left(n^{\log_2 3}\right)$$

(3) $T(n) = T(n/2) + n$

$$a < b^d$$

$$\Rightarrow T(n) = O(n)$$

MASTER'S THEOREM :

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

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WHERE $a \geq 1,\ b \geq 2,\ c \geq 0,\ d \geq 1$

THEN
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PROVE MASTER'S THEOREM

MASTER'S THEOREM:

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

$$T(1) = c$$

WHERE $a \geq 1,\ b \geq 2,\ c \geq 0,\ d \geq 1$

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PROOF: $$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

MASTER'S THEOREM:

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PROOF:
$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

$$= a\left(a\,T\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

$$= a^2\,T\left(\frac{n}{b^2}\right) + a\left(\frac{n}{b}\right)^d + n^d$$

MASTER'S THEOREM:

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

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WHERE $a \geq 1,\ b \geq 2,\ c \geq 0,\ d \geq 1$

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$$= a\left(a\,T\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

$$= a^2\,T\left(\frac{n}{b^2}\right) + a\left(\frac{n}{b}\right)^d + n^d$$

$$= a^2\left(a\,T\left(\frac{n}{b^3}\right) + \left(\frac{n}{b^2}\right)^d\right) + a\left(\frac{n}{b}\right)^d + n^d$$

MASTER'S THEOREM:

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

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PROOF: $T(n) = a\,T\left(\frac{n}{b}\right) + n^d$

$$= a\left(a\,T\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

$$= a^2\,T\left(\frac{n}{b^2}\right) + a\left(\frac{n}{b}\right)^d + n^d$$

$$= a^2\left(a\,T\left(\frac{n}{b^3}\right) + \left(\frac{n}{b^2}\right)^d\right) + a\left(\frac{n}{b}\right)^d + n^d$$

$$= a^3\,T\left(\frac{n}{b^3}\right) + a^2\left(\frac{n}{b^2}\right)^d + a\left(\frac{n}{b}\right)^d + n^d$$

$$\vdots$$

MASTER'S THEOREM:

LET $T(n)$ BE A MONOTONICALLY INCREASING FUNCTION THAT SATISFIES

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

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PROOF:
$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

$$= a\left(a\,T\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

$$= a^2\,T\left(\frac{n}{b^2}\right) + a\left(\frac{n}{b}\right)^d + n^d$$

$$= a^2\left(a\,T\left(\frac{n}{b^3}\right) + \left(\frac{n}{b^2}\right)^d\right) + a\left(\frac{n}{b}\right)^d + n^d$$

$$= a^3\,T\left(\frac{n}{b^3}\right) + a^2\left(\frac{n}{b^2}\right)^d + a\left(\frac{n}{b}\right)^d + n^d$$

$$\vdots$$

$$= a^k\,T\left(\frac{n}{b^k}\right) + a^{k-1}\left(\frac{n}{b^{k-1}}\right)^d + \dots + a\left(\frac{n}{b}\right)^d + n^d$$

PROOF:

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

$$= a\left(a\,T\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

$$= a^2\,T\left(\frac{n}{b^2}\right) + a\left(\frac{n}{b}\right)^d + n^d$$

$$= a^2\left(a\,T\left(\frac{n}{b^3}\right) + \left(\frac{n}{b^2}\right)^d\right) + a\left(\frac{n}{b}\right)^d + n^d$$

$$= a^3\,T\left(\frac{n}{b^3}\right) + a^2\left(\frac{n}{b^2}\right)^d + a\left(\frac{n}{b}\right)^d + n^d$$

$$\vdots$$

$$= a^k\,\underbrace{T\left(\frac{n}{b^k}\right)}_{=1} + a^{k-1}\left(\frac{n}{b^{k-1}}\right)^d + \cdots + a\left(\frac{n}{b}\right)^d + n^d$$

WHEN $\frac{n}{b^k} = 1$

$$\Rightarrow b^k = n$$

$$\Rightarrow k = \log_b n$$

PROOF: $T(n) = a\,T\left(\frac{n}{b}\right) + n^d$

$$= a\left(a\,T\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

$$= a^2\,T\left(\frac{n}{b^2}\right) + a\left(\frac{n}{b}\right)^d + n^d$$

$$= a^2\left(a\,T\left(\frac{n}{b^3}\right) + \left(\frac{n}{b^2}\right)^d\right) + a\left(\frac{n}{b}\right)^d + n^d$$

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$\vdots$

$k = \log_b n$ $\qquad = a^k\,T(1) + a^{k-1}\left(\frac{n}{b^{k-1}}\right)^d + \cdots + a\left(\frac{n}{b}\right)^d + n^d$

$$= a^k \cdot c + n^d\Big[$$

PROOF: $T(n) = a\,T\left(\frac{n}{b}\right) + n^d$

$$= a\left(a\,T\left(\frac{n}{b^2}\right) + \left(\frac{n}{b}\right)^d\right) + n^d$$

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$$= a^{\log_b n} \cdot c + n^d \sum_{i=1}^{\log_b n - 1}\left(\frac{a}{b^d}\right)^i$$

PROOF:

$$T(n) = a\,T\left(\frac{n}{b}\right) + n^d$$

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$$= a^{\log_b n} \cdot c + n^d \sum_{i=1}^{\log_b n - 1}\left(\frac{a}{b^d}\right)^i$$

$$= n^{\log_b a} \cdot c + n^d \sum_{i=1}^{\log_b n - 1}\left(\frac{a}{b^d}\right)^i$$

$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 1</u> : $a < b^d$

$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

CASE 1 : $a < b^d$

$\Rightarrow \log_b a < d$

$$T(n) = c \cdot \underbrace{n^{\log_b a}}_{< n^d} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

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$$\frac{a}{b^d} < 1$$

$$\Rightarrow \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i < \sum_{i=1}^{\infty} \left(\frac{a}{b^d}\right)^i$$

$$T(n) = c \cdot \underbrace{n^{\log_b a}}_{< n^d} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 1</u> : $a < b^d$

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$$\Rightarrow T(n) \le c \cdot n^d + c' \cdot n^d = O(n^d)$$

$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 2</u> :

$$a = b^d$$

$$\Rightarrow \log_b a = d$$

$$T(n) = c \cdot \underbrace{n^{\log_b a}}_{n^d} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 2</u> : $a = b^d$

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$$T(n) = c \cdot \underbrace{n^{\log_b a}}_{n^d} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 2</u> : $a = b^d$

$$\Rightarrow \log_b a = d$$

$$\sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i = \sum_{i=1}^{\log_b n - 1} (1)$$

$$T(n) = c \cdot \underbrace{n^{\log_b a}}_{n^d} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

CASE 2 : $\quad a = b^d$

$$\Rightarrow \quad \log_b a = d$$

$$\sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i = \sum_{i=1}^{\log_b n - 1} (1) \leq \log_b n$$

$$\Rightarrow \quad T(n) = c \cdot n^d + n^d \log_b n$$

$$= O\left(n^d \frac{\log_2 n}{\log_2 b}\right)$$

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$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 3</u> : $a > b^d$

$\Rightarrow \log_b a > d$

$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 3</u> : $a > b^d$

$$\Rightarrow \quad \log_b a > d$$

FIRST TERM REMAINS SAME

$$\sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i =$$

$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 3</u> : $\quad a > b^d$

$$\Rightarrow \quad \log_b a > d$$

FIRST TERM REMAINS SAME

$$\sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^a}\right)^i = \frac{\left(\frac{a}{b^a}\right)^{\log_b n} - 1}{\left(\frac{a}{b^a}\right) - 1}$$

$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

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$\Rightarrow \log_b a > d$

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$$\sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i = \frac{\left(\frac{a}{b^d}\right)^{\log_b n} - 1}{\left(\frac{a}{b^d}\right) - 1}$$

CONSTANT c'

$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

<u>CASE 3</u> : $a > b^d$

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$$\sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i = \frac{\left(\frac{a}{b^d}\right)^{\log_b n} - 1}{\left(\frac{a}{b^d}\right) - 1}$$

CONSTANT c'

$$\leq c' \left(\frac{a}{b^d}\right)^{\log_b n}$$

$$= c' \, n^{\log_b \left(\frac{a}{b^d}\right)}$$

$$= c' \left(n^{\log_b a - \log_b b^d}\right)$$

$$= c' \, n^{\log_b a - d}$$

$$= \frac{c' \, n^{\log_b a}}{n^d}$$

$$T(n) = c \cdot n^{\log_b a} + n^d \sum_{i=1}^{\log_b n - 1} \left(\frac{a}{b^d}\right)^i$$

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$$\leq c' \left(\frac{a}{b^d}\right)^{\log_b n}$$

$$= c' \, n^{\log_b \left(\frac{a}{b^d}\right)}$$

$$= c' \left(n^{\log_b a - \log_b b^d}\right)$$

$$= c' \, n^{\log_b a - d}$$

$$= \frac{c' \, n^{\log_b a}}{n^d}$$

$$T(n) = c \, n^{\log_b a} + c' n^{\log_b a} = O\left(n^{\log_b a}\right).$$